6 Appendix (For Online Publication)

Proof of Lemma 1. Suppose $((e_1^*, \theta_1^*, x_i^*), (e_2^*, \theta_2^*, x_2^*))$ is a pure-strategy Nash equilibrium. Then, e_i^* must satisfy: (1) $e_i^* \in \arg \max_{e_i} U_i((e_i, \theta_i^*, x_i^*), (e_j^*, \theta_j^*, x_j^*))$ and (2) $e_{i2}^* = 0$ whenever there exists $e_i \in \arg \max_{e_i} U_i((e_i, \theta_i^*, x_i^*), (e_j^*, \theta_j^*, x_j^*))$ with $e_{i2} = 0$. It follows from (1) and (2) that $e_{i1}^* = M_{i1}$ and $e_{i2}^* = 0$ if $M_{i1} \geq M_{i2}$, and $e_{i1}^* = 0$ and $e_{i2}^* = M_{i2}$ if $M_{i1} < M_{i2}$, where $M_{is} = \frac{\partial E_i}{\partial e_{is}}$. Furthermore, we obtain the following formulas: $M_{i1} = (\theta_{i1}^* + G(x_1^*, x_2^*) \cdot \theta_{j1}^*) \left(\frac{n+1}{n+2} \alpha_i\right)$ and $M_{i2} = (\theta_{i2}^* + G(x_1^*, x_2^*) \cdot \theta_{j2}^*) \left(\frac{n+1}{n+2}\right)$. $\qquad \square$

Proof of Lemma 2. Suppose $((e_1^*, \theta_1^*, x_i^*), (e_2^*, \theta_2^*, x_2^*))$ is a pure-strategy Nash equilibrium. Then, θ_{is}^* must satisfy: (1) $\theta_{is}^* \in \arg \max_{\theta_{is}} U_i((e_i^*, (\theta_{is}, \theta_{is'}^*), x_i^*), (e_j^*, \theta_j^*, x_j^*))$ and (2) $\theta_{is}^* = 0$ if $0 \in \arg \max_{\theta_{is}} U_i((e_i^*, (\theta_{is}, \theta_{is'}^*), x_i^*), (e_j^*, \theta_j^*, x_j^*))$. $1 \in \arg \max_{\theta_{is}} U_i((e_i^*, (\theta_{is}, \theta_{is'}^*), x_i^*), (e_j^*, \theta_j^*, x_j^*))$ if and only if $a_{is}^* - \overline{a}_s^* \geq 0$, and $0 \in \arg \max_{\theta_{is}} U_i((e_i^*, (\theta_{is}, \theta_{is'}^*), x_i^*), (e_j^*, \theta_j^*, x_j^*))$ if and only if $a_{is}^* - \overline{a}_s^* \leq 0$. It follows from (1) and (2) that $\theta_{is}^* = 1$ if and and only if $a_{is}^* - \overline{a}_s^* > 0$.

From Lemma 1, we know that, for some s, $e_{is}^* = 0$. When $e_{is}^* = 0$, $a_{is}^* - \overline{a}_s \leq 0$. Hence, $\theta_{is}^* = 0$. Therefore, $\theta_{is}^* = 0$ for some s. This completes the proof. $\qquad \square$

Proof of Lemma 3. Player i's self-esteem is given by: $E_i^i = \sum_{s=1}^2 \theta_{is}^*(a_{is}^* - \overline{a}_s^*)$. By Lemma 2, we know that $a_{is}^* - \overline{a}_s^* > 0$ if $\theta_{is}^* = 1$. It follows that $E_i^i \geq 0$ and $E_i^i > 0$ if $\theta_{i1}^* = 1$ or $\theta_{i2}^* = 1$. This proves (1).

Recall that $E_i^j = \sum_{s=1}^2 \theta_{js}^*(a_{is}^* - \overline{a}_s^*)$. If players hold the same values ($\theta_i^* = \theta_j^*$), it follows that $E_i^j = E_i^i$. Since $E_i^i \geq 0$, $E_i^j \geq 0$ when players hold the same values. Since $E_i^i > 0$ when player i values academics or music, $E_i^j > 0$ when the players hold the same values and value academics or music. This proves (2).

Now suppose players hold different values ($\theta_i^* \neq \theta_j^*$). Suppose $\theta_{js}^* = 1$. It is easy to show that $\theta_{is}^* = 0$. What if this were not the case and $\theta_{is}^* = 1$ instead? From Lemma 2, we know players value at most one activity. So, if $\theta_{is}^* = \theta_{js}^* = 1$, the players must hold the same values, which contradicts $\theta_i^* \neq \theta_j^*$. Since we know $\theta_{is}^* = 0$, it follows from Lemma 2 that $a_{is}^* - \overline{a}_s^* \leq 0$. Thus, whenever $\theta_{js}^* = 1$, $a_{is}^* - \overline{a}_s^* \leq 0$, it follows that $E_i^j \leq 0$. This proves (3). $\qquad \square$

Proof of Lemma 4. Suppose the players hold the same values and they value academics or music. It follows from Lemma 3 that the players have strictly positive esteem for one another ($E_i^j > 0$). If $k = 0^+$, the players must interact in equilibrium. We can prove this by contradiction. Suppose the players do not interact in equilibrium ($x_1^* = x_2^* = 0$). In equilibrium, $x_i^* \in \arg\max_{x_i} U_i((e_i^*, \theta_i^*, x_i), (e_j^*, \theta_j^*, x_j^*))$. But $0 \notin U_i((e_i^*, \theta_i^*, x_i), (e_j^*, \theta_j^*, x_j^*))$ since player i, in choosing $x_i = 1$ rather than $x_i = 0$ gains utility $E_i^j - k$. And, $E_i^j - k > 0$ since $E_i^j > 0$ and $k = 0^+$. This proves the first part of the lemma.

Suppose one player values academics and the other values music. It follows from Lemma 3 that the players negatively esteem one another ($E_i^j \leq 0$). The players will not interact in equilibrium if $k = 0^+$. Again, we can prove this by contradiction. Suppose the players do interact. Then, for one of the players (player i), we must have $x_i^* = 1$. In equilibrium, $x_i^* \in \arg\max_{x_i} U_i((e_i^*, \theta_i^*, x_i), (e_j^*, \theta_j^*, x_j^*))$. But $0 \notin U_i((e_i^*, \theta_i^*, x_i), (e_j^*, \theta_j^*, x_j^*))$. If player i chooses $x_i = 0$ rather than $x_i = 1$, he gains utility $k - (1 - x_j)E_i^j$. Since, $E_i^j \leq 0$ and $k = 0^+$, $k - (1 - x_j)E_i^j > 0$. This proves the second part of the lemma. $\qquad\square$

Proof of Proposition 1. Suppose $\alpha_2 \geq \alpha_1$ and $k = 0^+$. We will characterize the equilibria that exist as a function of α_1, α_2, and n.

It will be useful, in characterizing the equilibria, to use some shorthand. According to Lemma 1, we can denote an equilibrium by $((\theta_1^*, x_1^*), (\theta_2^*, x_2^*))$. It is not necessary to specify the effort level, since it can be deduced from Lemma 1. We also know from Lemma 2 that $\theta_i^* \in \{(1,0), (0,1), (0,0)\}$. Therefore, we will examine all combinations of the following form to see, under what parameters, they are equilibria of the game: $\{((\theta_1, x_1), (\theta_2, x_2)) : \theta_i \in \{(1,0), (0,1), (0,0)\}, x_i \in \{0,1\}\}$.

To check whether these combinations are equilibria, we will need to consider a variety of deviations by the players. We will denote deviations by player i by (θ_i', x_i'). We do not bother to specify the effort level of player i when he deviates, since we know the optimal choice of effort. It is optimal for player i, when he deviates, to choose $e_{i1}' = M_{i1}$ and $e_{i2}' = 0$ if $M_{i1} \geq M_{i2}$; it is optimal for player i to choose $e_{i1}' = 0$ and $e_{i2}' = M_{i2}$ if $M_{i1} < M_{i2}$; where $M_{i1} = (\theta_{i1}' + G(x_i', x_j) \cdot \theta_{j1}) \left(\frac{n+1}{n+2}\alpha_i\right)$ and $M_{i2} = (\theta_{i2}' + G(x_i', x_j) \cdot \theta_{j2}) \left(\frac{n+1}{n+2}\right)$. Player i always focuses effort on one activity when he deviates and so is below average

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at one of the activities. Thus, he will never choose to deviate to $\theta'_i = (1, 1)$. Thus, we will only check deviations of the form: $\{(\theta'_i, x'_i) : \theta_i \in \{(1,0),(0,1),(0,0)\}, x_i \in \{0,1\}\}$.

We will now systematically check all the combinations. We start with cases where players do not interact ($x_1 = x_2 = 0$) and then turn to cases where players do interact.

Case 1: No interaction ($x_1 = x_2 = 0$)

There are five types of combinations to check. We will consider each in turn.

1. $\theta_1 = \theta_2 = (1, 0)$

   If the players follow these strategies, $U_2 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_2^2 - \frac{n+1}{(n+2)^2}\alpha_1^2$. If player 2 deviates to $\theta'_2 = (1,0)$, $x'_2 = 1$, this yields: $U'_2 = 2\left(\frac{n+1}{n+2}\right)^2 \alpha_2^2 - 2\frac{n+1}{(n+2)^2}\alpha_1^2$. This deviation is profitable if $\frac{3}{2}(n+1)\alpha_2^2 > \alpha_1^2$. Suppose this deviation is not profitable: $\frac{3}{2}(n+1)\alpha_2^2 \le \alpha_1^2$. If player 2 instead deviates to $\theta'_2 = (0,0)$, $x'_2 = 0$, this yields: $U'_2 = 0$. This deviation is profitable if $\frac{1}{2}(n+1)\alpha_2^2 \le \alpha_1^2$ (which is true, since $\frac{3}{2}(n+1)\alpha_2^2 \le \alpha_1^2$). In summary, one of these two deviations will be profitable. Thus, (1) is never an equilibrium.

2. $\theta_1 = (1,0), \theta_2 = (0,1)$

   If the players follow these strategies, they receive: $U_1 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_2^2$, $U_2 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2$. In order for this to be an equilibrium, there cannot be any profitable deviations. There are ten deviations we need to check.

   (i) Player 1 deviates to $\theta'_1 = (0,1)$, $x'_1 = 0$: yields $U'_1 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - \frac{n+1}{(n+2)^2}$.

   (ii) Player 1 deviates to $\theta'_1 = (0,0)$, $x'_1 = 0$: yields $U'_1 = 0$.

   (iii) Player 1 deviates to $\theta'_1 = (1,0)$, $x'_1 = 1$: yields $U'_1 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \max(\alpha_1^2, 1) - \frac{n+1}{(n+2)^2}$.

   (iv) Player 1 deviates to $\theta'_1 = (0,1)$, $x'_1 = 1$: yields $U'_1 = 2\left(\frac{n+1}{n+2}\right)^2 - 2\frac{n+1}{(n+2)^2}$.

   (v) Player 1 deviates to $\theta'_1 = (0,0)$ and $x'_1 = 1$: yields $U'_1 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - \frac{n+1}{(n+2)^2}$.

   (vi) Player 2 deviates to $\theta'_2 = (1,0)$, $x'_2 = 0$: yields $U'_2 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_2^2 - \frac{n+1}{(n+2)^2}\alpha_1^2$.

   (vii) Player 2 deviates to $\theta'_2 = (0,0)$, $x'_2 = 0$: yields $U'_2 = 0$.

   (viii) Player 2 deviates to $\theta'_2 = (1,0)$, $x'_2 = 1$: yields $U'_2 = 2\left(\frac{n+1}{n+2}\right)^2 \alpha_2^2 - 2\frac{n+1}{(n+2)^2}\alpha_1^2$.

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(ix) Player 2 deviates to $\theta_2' = (0,1)$, $x_2' = 1$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \max(\alpha_2^2, 1) - \frac{n+1}{(n+2)^2}\alpha_1^2$.

(x) Player 2 deviates to $\theta_2' = (0,0)$ and $x_2' = 1$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_2^2 - \frac{n+1}{(n+2)^2}\alpha_1^2$.

Observe that (2) will be an equilibrium if and only if (i) through (x) are not profitable deviations. This yields ten conditions for (2) to be an equilibrium, but only conditions (iv) and (viii) are binding:

Condition (iv): $\alpha_1^2 \geq \frac{4n}{n+1}$

Condition (viii): $\alpha_1^2 \geq (n+1)(\alpha_2^2 - \frac{1}{4})$

These conditions can be simultaneously met if $n = 0$ but not when $n \geq 1$. Let us see why. Since $\alpha_1^2 \leq \alpha_2^2$, we must have: $(n+1)(\alpha_2^2 - \frac{1}{4}) \leq \alpha_2^2$, or $\alpha_2^2 \leq \frac{n+1}{4n}$. This implies we must also have $\alpha_1^2 \leq \frac{n+1}{4n}$. Since we also have $\alpha_1^2 \geq \frac{4n}{n+1}$, it follows that we must have: $\frac{4n}{n+1} \leq \alpha_1^2 \leq \frac{n+1}{4n}$. For $n \geq 1$, $\frac{4n}{n+1} > \frac{n+1}{4n}$, so equilibria of this type will not exist.

3. $\theta_1 = (0,1), \theta_2 = (1,0)$

If the players follow these strategies, they receive: $U_1 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2$, $U_2 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_2^2$. In order for this to be an equilibrium, there cannot be any profitable deviations. There are ten deviations we need to check.

(i) Player 1 deviates to $\theta_1' = (1,0)$, $x_1' = 0$: yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_1^2 - \frac{n+1}{(n+2)^2}\alpha_2^2$.

(ii) Player 1 deviates to $\theta_1' = (0,0)$, $x_1' = 0$: yields $U_1' = 0$.

(iii) Player 1 deviates to $\theta_1' = (1,0)$, $x_1' = 1$: yields $U_1' = 2\left(\frac{n+1}{n+2}\right)^2 \alpha_1^2 - 2\frac{n+1}{(n+2)^2}\alpha_2^2$.

(iv) Player 1 deviates to $\theta_1' = (0,1)$, $x_1' = 1$: yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \max(\alpha_1^2, 1) - \frac{n+1}{(n+2)^2}\alpha_2^2$.

(v) Player 1 deviates to $\theta_1' = (0,0)$ and $x_1' = 1$: yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_1^2 - \frac{n+1}{(n+2)^2}\alpha_2^2$.

(vi) Player 2 deviates to $\theta_2' = (0,1)$, $x_2' = 0$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - \frac{n+1}{(n+2)^2}$.

(vii) Player 2 deviates to $\theta_2' = (0,0)$, $x_2' = 0$: yields $U_2' = 0$.

(viii) Player 2 deviates to $\theta_2' = (1,0)$, $x_2' = 1$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_2^2 - \frac{n+1}{(n+2)^2}$.

(ix) Player 2 deviates to $\theta_2' = (0,1)$, $x_2' = 1$: yields $U_2' = 2\left(\frac{n+1}{n+2}\right)^2 - 2\frac{n+1}{(n+2)^2}$.

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(x) Player 2 deviates to $\theta'_2 = (0,0)$ and $x'_2 = 1$: yields $U'_2 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - \frac{n+1}{(n+2)^2}$.

Observe that (3) will be an equilibrium if and only if (i) through (x) are not profitable deviations. This yields ten conditions for (3) to be an equilibrium, but only conditions (iii)/(v) and (ix) are binding:

Condition (iii)/(v): $\alpha_2^2 \geq (n+1)(\alpha_1^2 - \frac{1}{4})$

Condition (ix): $\alpha_2^2 \geq \frac{4n}{n+1}$

Combining these conditions, we obtain: $\alpha_2^2 \geq \max\left(\frac{4n}{n+1}, (n+1)(\alpha_1^2 - \frac{1}{4})\right)$

4. $\theta_1 = (0,1), \theta_2 = (0,1)$

   If the players follow these strategies, $U_2 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - \frac{n+1}{(n+2)^2}$. If player 2 deviates to $\theta'_2 = (0,1)$, $x'_2 = 1$, this yields: $U'_2 = 2\left(\frac{n+1}{n+2}\right)^2 - 2\frac{n+1}{(n+2)^2}$. This deviation is always profitable. Thus, (4) is never an equilibrium.

5. $\theta_1 = (0,0)$ **or** $\theta_2 = (0,0)$

   Suppose $\theta_i = (0,0)$ and player j chooses some θ_j. Then, when the players follow these strategies $U_i = 0$. We know that player j focuses on at most one activity in equilibrium. Suppose player j is not focused on activity 1. If player i deviates to $\theta'_{is} = (1,0)$, $x'_i = 0$, this yields $U'_i = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_i^2 > 0$. Suppose instead player j is not focused on activity 2. If player i deviates to $\theta'_{is} = (0,1)$, $x'_i = 0$, this yields $U'_i = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 > 0$. So, there is always a profitable deviation. Thus, no equilibria of this type exist.

Case 2a: Interaction, initiated by both players ($x_1 = x_2 = 1$)

Observe that there will be a profitable deviation for player 1 to $\theta'_1 = \theta_1$, $x'_1 = 0$ since $k > 0$. Thus, no equilibria exist in which both players initiate interaction.

Case 2b: Interaction, initiated by player 1 only ($x_1 = 1, x_2 = 0$)

Observe that, because $k > 0$, it will not be profitable for player 2 to deviate to $x'_2 = 1$. Thus, it will be sufficient to restrict attention to deviations by player 2 in which $x'_2 = 0$.

We know from Lemma 4 that, since $k = 0^+$, equilibria do not exist in which $\theta_i = (1, 0)$, $\theta_j = (0, 1)$, and players interact. This means that there are three types of combinations to check.

1. $\theta_1 = \theta_2 = (1, 0)$

   If the players follow these strategies, they receive: $U_1 = 2 \left(\frac{n+1}{n+2}\right)^2 \alpha_1^2 - 4\frac{n+1}{(n+2)^2}\alpha_2^2$, $U_2 = 2 \left(\frac{n+1}{n+2}\right)^2 \alpha_2^2 - 4\frac{n+1}{(n+2)^2}\alpha_1^2$. In order for this to be an equilibrium, there cannot be any profitable deviations. There are seven deviations we need to check.

   (i) Player 1 deviates to $\theta_1' = (0, 1)$, $x_1' = 1$: yields $U_1' = \frac{1}{2} \left(\frac{n+1}{n+2}\right)^2 \max(\alpha_1^2, 1) - 2\frac{n+1}{(n+2)^2}\alpha_2^2$.

   (ii) Player 1 deviates to $\theta_1' = (0, 0)$, $x_1' = 1$: yields $U_1' = \frac{1}{2} \left(\frac{n+1}{n+2}\right)^2 \alpha_1^2 - 2\frac{n+1}{(n+2)^2}\alpha_2^2$.

   (iii) Player 1 deviates to $\theta_1' = (1, 0)$, $x_1' = 0$: yields $U_1' = \frac{1}{2} \left(\frac{n+1}{n+2}\right)^2 \alpha_1^2 - 2\frac{n+1}{(n+2)^2}\alpha_2^2$.

   (iv) Player 1 deviates to $\theta_1' = (0, 1)$, $x_1' = 0$: yields $U_1' = \frac{1}{2} \left(\frac{n+1}{n+2}\right)^2$.

   (v) Player 1 deviates to $\theta_1' = (0, 0)$, $x_1' = 0$: yields $U_1' = 0$.

   (vi) Player 2 deviates to $\theta_2' = (0, 1)$, $x_2' = 0$: yields $U_2' = \frac{1}{2} \left(\frac{n+1}{n+2}\right)^2 \max(\alpha_2^2, 1) - 2\frac{n+1}{(n+2)^2}\alpha_1^2$.

   (vii) Player 2 deviates to $\theta_2' = (0, 0)$, $x_2' = 0$: yields $U_2' = \frac{1}{2} \left(\frac{n+1}{n+2}\right)^2 \alpha_2^2 - 2\frac{n+1}{(n+2)^2}\alpha_1^2$.

   Observe that (1) will be an equilibrium if and only if (i) through (vii) are not profitable deviations. This yields seven conditions for (1) to be an equilibrium, but only conditions (iv), (vi), and (vii) are binding:

   Condition (iv): $\alpha_2^2 < \frac{1}{2}(n + 1)(\alpha_1^2 - \frac{1}{4})$ (this inequality is strict because player 1 prefers not to initiate interaction, since $k = 0^+$).

   Condition (vi): $\alpha_2^2 \geq \frac{\alpha_1^2}{n+1} + \frac{1}{4}$

   Condition (vii): $\alpha_2^2 \geq \frac{4}{3}\frac{\alpha_1^2}{n+1}$

2. $\theta_1 = (0, 1), \theta_2 = (0, 1)$

   If the players follow these strategies, they receive: $U_1 = 2 \left(\frac{n+1}{n+2}\right)^2 - 4\frac{n+1}{(n+2)^2}$, $U_2 = 2 \left(\frac{n+1}{n+2}\right)^2 - 4\frac{n+1}{(n+2)^2}$. In order for this to be an equilibrium, there cannot be any profitable deviations. There are seven deviations we need to check.

(i) Player 1 deviates to $\theta_1' = (1,0)$, $x_1' = 1$: yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \max(\alpha_1^2, 1) - 2\frac{n+1}{(n+2)^2}$.

(ii) Player 1 deviates to $\theta_1' = (0,0)$, $x_1' = 1$: yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - 2\frac{n+1}{(n+2)^2}$.

(iii) Player 1 deviates to $\theta_1' = (1,0)$, $x_1' = 0$: yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_1^2$.

(iv) Player 1 deviates to $\theta_1' = (0,1)$, $x_1' = 0$: yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - 2\frac{n+1}{(n+2)^2}$.

(v) Player 1 deviates to $\theta_1' = (0,0)$, $x_1' = 0$: yields $U_1' = 0$.

(vi) Player 2 deviates to $\theta_2' = (1,0)$, $x_2' = 0$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \max(\alpha_2^2, 1) - 2\frac{n+1}{(n+2)^2}$.

(vii) Player 2 deviates to $\theta_2' = (0,0)$, $x_2' = 0$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - 2\frac{n+1}{(n+2)^2}$.

Observe that (2) will be an equilibrium if and only if (i) through (vii) are not profitable deviations. This yields seven conditions for (2) to be an equilibrium, but only conditions (iii) and (vi) are binding:

Condition (iii): $\alpha_1^2 < 4\left(\frac{n-1}{n+1}\right)$ (this inequality is strict because player 1 prefers not to initiate interaction, since $k = 0^+$).

Condition (vi): $\alpha_2^2 \leq 4\left(\frac{n}{n+1}\right)$

3. $\theta_1 = (0,0)$ **or** $\theta_2 = (0,0)$

Suppose $\theta_2 = (0,0)$. It is always profitable for player 1 to deviate to $x_1' = 0$. Suppose $\theta_1 = (0,0)$ and $\theta_2 = (1,0)$. Then $U_1 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_1^2 - \frac{n+1}{(n+2)^2}\alpha_2^2$. If player 1 deviates to $x_1' = 0$, this yields $U_1' = 0$ (which is profitable when $\alpha_2^2 \geq \frac{1}{2}(n+1)\alpha_1^2$). If player 1 instead deviates to $\theta_1' = (1,0)$, this yields $U_1' = 4\left(\frac{n+1}{n+2}\right)^2 \alpha_1^2 - 2\frac{n+1}{(n+2)^2}\alpha_2^2$ (which is profitable when $\frac{3}{2}(n+1)\alpha_1^2 \geq \alpha_2^2$). Clearly, one of these deviations will be profitable. Finally, suppose $\theta_1 = (0,0)$ and $\theta_2 = (0,1)$. Then $U_1 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - \frac{n+1}{(n+2)^2}$. If player 1 instead deviates to $\theta_1' = (0,1)$, this yields $U_1' = 2\left(\frac{n+1}{n+2}\right)^2 - 2\frac{n+1}{(n+2)^2}$, which is always profitable. Hence, there is always a profitable deviation when $\theta_1 = (0,0)$ or $\theta_2 = (0,0)$.

Case 2c: Interaction, initiated by player 2 only ($x_1 = 0, x_2 = 1$)

Observe that, because $k > 0$, it will not be profitable for player 2 to deviate to $x_2' = 1$. Thus, it will be sufficient to restrict attention to deviations by player 2 in which $x_2' = 0$.

We know from Lemma 4 that, since $k = 0^+$, equilibria do not exist in which $\theta_i = (1, 0)$, $\theta_j = (0, 1)$, and players interact. This means that there are three types of combinations to check.

1. $\theta_1 = \theta_2 = (1, 0)$

   If the players follow these strategies, they receive: $U_1 = 2\left(\frac{n+1}{n+2}\right)^2 \alpha_1^2 - 4\frac{n+1}{(n+2)^2}\alpha_2^2$, $U_2 = 2\left(\frac{n+1}{n+2}\right)^2 \alpha_2^2 - 4\frac{n+1}{(n+2)^2}\alpha_1^2$. In order for this to be an equilibrium, there cannot be any profitable deviations. There are seven deviations we need to check.

   (i) Player 1 deviates to $\theta_1' = (0, 1)$, $x_1' = 0$: yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \max(\alpha_1^2, 1) - 2\frac{n+1}{(n+2)^2}\alpha_2^2$.

   (ii) Player 1 deviates to $\theta_1' = (0, 0)$, $x_1' = 0$: yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_1^2 - 2\frac{n+1}{(n+2)^2}\alpha_2^2$.

   (iii) Player 2 deviates to $\theta_2' = (0, 1)$, $x_2' = 1$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \max(\alpha_2^2, 1) - 2\frac{n+1}{(n+2)^2}\alpha_1^2$.

   (iv) Player 2 deviates to $\theta_2' = (0, 0)$, $x_2' = 1$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_2^2 - 2\frac{n+1}{(n+2)^2}\alpha_1^2$.

   (v) Player 2 deviates to $\theta_2' = (1, 0)$, $x_2' = 0$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_2^2 - 2\frac{n+1}{(n+2)^2}\alpha_1^2$.

   (vi) Player 2 deviates to $\theta_2' = (0, 1)$, $x_2' = 0$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2$.

   (vii) Player 2 deviates to $\theta_2' = (0, 0)$, $x_2' = 0$: yields $U_2' = 0$.

   Observe that (1) will be an equilibrium if and only if (i) through (vii) are not profitable deviations. This yields seven conditions for (1) to be an equilibrium, but only conditions (i), (ii), and (vi) are binding:

   Condition (i): $\alpha_2^2 \leq (n+1)(\alpha_1^2 - \frac{1}{4})$.

   Condition (ii): $\alpha_2^2 < \frac{3}{4}(n+1)\alpha_1^2$ (this inequality is strict because player 1 prefers, all else equal, not to value activities).

   Condition (vi): $\alpha_2^2 > \frac{2\alpha_1^2}{n+1} + \frac{1}{4}$ (this inequality is strict because player 1 prefers not to initiate interaction, since $k = 0^+$).

2. $\theta_1 = (0, 1), \theta_2 = (0, 1)$

   If the players follow these strategies, they receive: $U_1 = 2\left(\frac{n+1}{n+2}\right)^2 - 4\frac{n+1}{(n+2)^2}$, $U_2 = 2\left(\frac{n+1}{n+2}\right)^2 - 4\frac{n+1}{(n+2)^2}$. In order for this to be an equilibrium, there cannot be any profitable deviations. There are seven deviations we need to check.

(i) Player 1 deviates to $\theta'_1 = (1,0)$, $x'_1 = 0$: yields $U'_1 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \max(\alpha_1^2, 1) - 2\frac{n+1}{(n+2)^2}$.

(ii) Player 1 deviates to $\theta'_1 = (0,0)$, $x'_1 = 0$: yields $U'_1 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - 2\frac{n+1}{(n+2)^2}$.

(iii) Player 2 deviates to $\theta'_2 = (1,0)$, $x'_2 = 1$: yields $U'_2 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \max(\alpha_2^2, 1) - 2\frac{n+1}{(n+2)^2}$.

(iv) Player 2 deviates to $\theta'_2 = (0,0)$, $x'_2 = 1$: yields $U'_2 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - 2\frac{n+1}{(n+2)^2}$.

(v) Player 2 deviates to $\theta'_2 = (1,0)$, $x'_2 = 0$: yields $U'_2 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_2^2$.

(vi) Player 2 deviates to $\theta'_2 = (0,1)$, $x'_2 = 0$: yields $U'_2 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - 2\frac{n+1}{(n+2)^2}$.

(vii) Player 2 deviates to $\theta'_2 = (0,0)$, $x'_2 = 0$: yields $U'_2 = 0$.

Observe that (2) will be an equilibrium if and only if (i) through (vii) are not profitable deviations. This yields seven conditions for (2) to be an equilibrium, but only condition (v) is binding:

Condition (v): $\alpha_2^2 < 4\left(\frac{n-1}{n+1}\right)$ (this inequality is strict because player 2 prefers not to initiate interaction, since $k = 0^+$).

3. $\theta_1 = (0,0)$ **or** $\theta_2 = (0,0)$

Suppose $\theta_1 = (0,0)$. It is always profitable for player 2 to deviate to $x'_2 = 0$. Suppose $\theta_2 = (0,0)$ and $\theta_1 = (1,0)$. Then $U_2 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_2^2 - \frac{n+1}{(n+2)^2}\alpha_1^2$. If player 2 deviates to $x'_2 = 0$, this yields $U'_2 = 0$ (which is profitable when $\alpha_1^2 \geq \frac{1}{2}(n+1)\alpha_2^2$). If player 2 instead deviates to $\theta'_2 = (1,0)$, this yields $U'_2 = 4\left(\frac{n+1}{n+2}\right)^2 \alpha_2^2 - 2\frac{n+1}{(n+2)^2}\alpha_1^2$ (which is profitable when $\frac{3}{2}(n+1)\alpha_2^2 \geq \alpha_1^2$). Clearly, one of these deviations will be profitable. Finally, suppose $\theta_2 = (0,0)$ and $\theta_1 = (0,1)$. Then $U_2 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - \frac{n+1}{(n+2)^2}$. If player 2 instead deviates to $\theta'_2 = (0,1)$, this yields $U'_2 = 2\left(\frac{n+1}{n+2}\right)^2 - 2\frac{n+1}{(n+2)^2}$, which is always profitable. Hence, there is always a profitable deviation when $\theta_1 = (0,0)$ or $\theta_2 = (0,0)$.

Case 2: Combined

If we combine Cases 2a, 2b, and 2c we find the following.

1. An equilibrium with $\theta_1 = \theta_2 = (1,0)$ and interaction between the players exists if:

(i) $\alpha_2^2 > \frac{2\alpha_1^2}{n+1} + \frac{1}{4}$

(ii) $\alpha_2^2 \leq (n+1)(\alpha_1^2 - \frac{1}{4})$.

(iii) $\alpha_2^2 < \frac{3}{4}(n+1)\alpha_1^2$

These conditions are the same as those from case 2c, which are weaker than those in case 2b. Intuitively, player 2, who is better at academics than player 1, is more willing to initiate interaction than player 1.

2. An equilibrium with $\theta_1 = \theta_2 = (0,1)$ and interaction between the players exists if:

(i) $\alpha_1^2 < 4(\frac{n-1}{n+1})$

(ii) $\alpha_2^2 \leq 4(\frac{n}{n+1})$

These conditions are the same as those from case 2b, which are weaker than those in case 2c. Intuitively, player 1 is more willing to initiate interaction than player 2 in this instance.

This completes the proof. □

Proof of Proposition 2. Suppose $\alpha_2 \geq \alpha_1$ and $\alpha_2 > \overline{\alpha}_H$. We will characterize the equilibria that exist as a function of α_1, α_2, k, and n.

It will be useful, in characterizing the equilibria, to use the same shorthand as we used in the proof of Proposition 1. Once again, we will examine all combinations of the following form to see, under what parameters, they are equilibria of the game: $\{((\theta_1, x_1), (\theta_2, x_2)) : \theta_i \in \{(1,0), (0,1), (0,0)\}, x_i \in \{0,1\}\}$. We will first examine combinations in which player 2 is a scholar ($\theta_2 = (1,0)$). Later on, we will show that no equilibria exist in which player 2 is not a scholar.

Case 1: No interaction ($x_1 = x_2 = 0$)

Since we assume player 2 is a scholar ($\theta_2 = (1,0)$), we will denote combinations by player 1's choice of θ_1. There are three combinations to check.

1. $\theta_1 = (1, 0)$

   If the players follow these strategies, they receive: $U_1 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_1^2 - \frac{n+1}{(n+2)^2}\alpha_2^2$,
   $U_2 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_2^2 - \frac{n+1}{(n+2)^2}\alpha_1^2$. In order for this to be an equilibrium, there cannot be any profitable deviations. There are ten deviations we need to check.

   (i) Player 1 deviates to $\theta_1' = (0, 1)$, $x_1' = 0$: yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2$.

   (ii) Player 1 deviates to $\theta_1' = (0, 0)$, $x_1' = 0$: yields $U_1' = 0$.

   (iii) Player 1 deviates to $\theta_1' = (1, 0)$, $x_1' = 1$: yields $U_1' = 2\left(\frac{n+1}{n+2}\right)^2 \alpha_1^2 - 2\frac{n+1}{(n+2)^2}\alpha_2^2 - k$.

   (iv) Player 1 deviates to $\theta_1' = (0, 1)$, $x_1' = 1$: yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \max(\alpha_1^2, 1) - \frac{n+1}{(n+2)^2}\alpha_2^2 - k$.

   (v) Player 1 deviates to $\theta_1' = (0, 0)$, $x_1' = 1$: yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_1^2 - \frac{n+1}{(n+2)^2}\alpha_2^2 - k$.

   (vi) Player 2 deviates to $\theta_2' = (0, 1)$, $x_2' = 0$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2$.

   (vii) Player 2 deviates to $\theta_2' = (0, 0)$, $x_2' = 0$: yields $U_2' = 0$.

   (viii) Player 2 deviates to $\theta_2' = (1, 0)$, $x_2' = 1$: yields $U_2' = 2\left(\frac{n+1}{n+2}\right)^2 \alpha_2^2 - 2\frac{n+1}{(n+2)^2}\alpha_1^2 - k$.

   (ix) Player 2 deviates to $\theta_2' = (0, 1)$, $x_2' = 1$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_2^2 - \frac{n+1}{(n+2)^2}\alpha_1^2 - k$.

   (x) Player 2 deviates to $\theta_2' = (0, 0)$, $x_2' = 1$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_2^2 - \frac{n+1}{(n+2)^2}\alpha_1^2 - k$.

   Observe that (1) will be an equilibrium if and only if (i) through (x) are not profitable deviations. This yields ten conditions for (1) to be an equilibrium, but only conditions (i) and (viii) are binding:

   Condition (i): $\alpha_1^2 \geq 1 + \frac{2}{(n+1)}\alpha_2^2$.

   Condition (viii): $k \geq \left(\frac{n+1}{n+2}\right)^2 \left(\frac{3}{2}\alpha_2^2 - \frac{1}{n+1}\alpha_1^2\right) = \overline{k}_2$.

2. $\theta_1 = (0, 1)$

   If the players follow these strategies, they receive: $U_1 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2$, $U_2 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_2^2$. In order for this to be an equilibrium, there cannot be any profitable deviations. There are ten deviations we need to check.

   (i) Player 1 deviates to $\theta_1' = (1, 0)$, $x_1' = 0$: yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_1^2 - \frac{n+1}{(n+2)^2}\alpha_2^2$.

   (ii) Player 1 deviates to $\theta_1' = (0, 0)$, $x_1' = 0$: yields $U_1' = 0$.

(iii) Player 1 deviates to $\theta_1' = (1,0)$, $x_1' = 1$: yields $U_1' = 2\left(\frac{n+1}{n+2}\right)^2 \alpha_1^2 - 2\frac{n+1}{(n+2)^2}\alpha_2^2 - k$.

(iv) Player 1 deviates to $\theta_1' = (0,1)$, $x_1' = 1$: yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \max(1, \alpha_1^2) - \frac{n+1}{(n+2)^2}\alpha_2^2 - k$.

(v) Player 1 deviates to $\theta_1' = (0,0)$, $x_1' = 1$: yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_1^2 - \frac{n+1}{(n+2)^2}\alpha_2^2 - k$.

(vi) Player 2 deviates to $\theta_2' = (0,1)$, $x_2' = 0$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - \frac{n+1}{(n+2)^2}$.

(vii) Player 2 deviates to $\theta_2' = (0,0)$, $x_2' = 0$: yields $U_2' = 0$.

(viii) Player 2 deviates to $\theta_2' = (1,0)$, $x_2' = 1$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_2^2 - \frac{n+1}{(n+2)^2} - k$.

(ix) Player 2 deviates to $\theta_2' = (0,1)$, $x_2' = 1$: yields $U_2' = 2\left(\frac{n+1}{n+2}\right)^2 - 2\frac{n+1}{(n+2)^2} - k$.

(x) Player 2 deviates to $\theta_2' = (0,0)$, $x_2' = 1$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - \frac{n+1}{(n+2)^2} - k$.

Observe that (2) will be an equilibrium if and only if (i) through (x) are not profitable deviations. This yields ten conditions for (2) to be an equilibrium, but only conditions (i), (iii), and (viii) are binding:

Condition (i): $\alpha_1^2 \leq 1 + \frac{2}{n+1}\alpha_2^2$.

Condition (iii): $k \geq \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 [4\alpha_1^2 - \frac{4}{n+1}\alpha_2^2 - 1]$.

Condition (viii): $k \geq \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 [-\frac{2}{n+1}]$.

We can combine conditions (iii) and (viii) as follows:

$k \geq \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \max\left(4\alpha_1^2 - \frac{4}{n+1}\alpha_2^2 - 1, -\frac{2}{n+1}\right) = \overline{k}_1$.

3. $\theta_1 = (0,0)$

If the players follow these strategies, $U_1 = 0$. If player 1 deviates to $\theta_1' = (0,1)$, $x_1' = 0$, this yields: $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2$. So, there is always a profitable deviation. Thus, (3) is never an equilibrium.

Case 2a: Interaction, initiated by both players ($x_1 = x_2 = 1$)

Observe that there will be a profitable deviation to $x_i' = 0$ if $k > 0$. If $k \leq 0$, deviating to $x_i' = 0$ is (weakly) unprofitable. Therefore, it will be sufficient, after imposing the condition $k \leq 0$, to restrict attention to deviations in which $x_i' = 1$. There are three combinations to check.

1. $\theta_1 = (1, 0)$

   If the players follow these strategies, they receive: $U_1 = 2 \left(\frac{n+1}{n+2}\right)^2 \alpha_1^2 - 4\frac{n+1}{(n+2)^2}\alpha_2^2 - k$, $U_2 = 2 \left(\frac{n+1}{n+2}\right)^2 \alpha_2^2 - 4\frac{n+1}{(n+2)^2}\alpha_1^2 - k$. In order for this to be an equilibrium, there cannot be any profitable deviations. There are four deviations we need to check.

   (i) Player 1 deviates to $\theta_1' = (0,1)$, $x_1' = 1$: yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \max(\alpha_1^2, 1) - 2\frac{n+1}{(n+2)^2}\alpha_2^2 - k$.

   (ii) Player 1 deviates to $\theta_1' = (0,0)$, $x_1' = 1$: yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_1^2 - 2\frac{n+1}{(n+2)^2}\alpha_2^2 - k$.

   (iii) Player 2 deviates to $\theta_2' = (0,1)$, $x_2' = 1$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_2^2 - 2\frac{n+1}{(n+2)^2}\alpha_1^2 - k$.

   (iv) Player 2 deviates to $\theta_2' = (0,0)$, $x_2' = 1$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_2^2 - 2\frac{n+1}{(n+2)^2}\alpha_1^2 - k$.

   Observe that (1) will be an equilibrium if and only if (i) through (iv) are not profitable deviations. This yields four conditions for (1) to be an equilibrium, but only conditions (i) and (ii) are binding:

   Condition (i): $\alpha_1^2 \geq \frac{1}{4} + \frac{1}{n+1}\alpha_2^2$

   Condition (ii): $\alpha_1^2 > \frac{4}{3(n+1)}\alpha_2^2$ (This inequality is strict because player 1 prefers, all else equal, to not value activities.)

   Condition (k): $k \leq 0$

2. $\theta_1 = (0, 1)$

   If $\alpha_1^2 \geq 1$, there will always be a profitable deviation for player 1 to $\theta_1' = (0,0)$, $x_1' = 1$. So we assume $\alpha_1^2 < 1$. If the players follow these strategies, they receive: $U_1 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - \frac{n+1}{(n+2)^2}\alpha_2^2 - k$, $U_2 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_2^2 - \frac{n+1}{(n+2)^2} - k$. In order for this to be an equilibrium, there cannot be any profitable deviations. There are four deviations we need to check.

   (i) Player 1 deviates to $\theta_1' = (1,0)$, $x_1' = 1$: yields $U_1' = 2\left(\frac{n+1}{n+2}\right)^2 \alpha_1^2 - 2\frac{n+1}{(n+2)^2}\alpha_2^2 - k$.

   (ii) Player 1 deviates to $\theta_1' = (0,0)$, $x_1' = 1$: yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_1^2 - \frac{n+1}{(n+2)^2}\alpha_2^2 - k$.

   (iii) Player 2 deviates to $\theta_2' = (0,1)$, $x_2' = 1$: yields $U_2' = 2\left(\frac{n+1}{n+2}\right)^2 - 2\frac{n+1}{(n+2)^2} - k$.

   (iv) Player 2 deviates to $\theta_2' = (0,0)$, $x_2' = 1$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - \frac{n+1}{(n+2)^2} - k$.

Observe that (2) will be an equilibrium if and only if (i) through (iv) are not profitable deviations. This yields four conditions for (2) to be an equilibrium, but only conditions (i) and (ii) are binding:

Condition (i): $\alpha_1^2 \leq \frac{1}{4} + \frac{1}{2(n+1)}\alpha_2^2$

Condition (ii): $\alpha_1^2 < 1$ (This inequality is strict because player 1 prefers, all else equal, to not value activities.)

Condition (k): $k \leq 0$

3. $\theta_1 = (0,0)$

If the players follow these strategies, they receive: $U_1 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_1^2 - \frac{n+1}{(n+2)^2}\alpha_2^2 - k$, $U_2 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_2^2 - \frac{n+1}{(n+2)^2}\alpha_1^2 - k$.

In order for this to be an equilibrium, there cannot be any profitable deviations. There are four deviations we need to check.

(i) Player 1 deviates to $\theta_1' = (1,0)$, $x_1' = 1$: yields $U_1' = 2\left(\frac{n+1}{n+2}\right)^2 \alpha_1^2 - 2\frac{n+1}{(n+2)^2}\alpha_2^2 - k$.

(ii) Player 1 deviates to $\theta_1' = (0,1)$, $x_1' = 1$: yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \max(\alpha_1^2, 1) - \frac{n+1}{(n+2)^2}\alpha_2^2 - k$.

(iii) Player 2 deviates to $\theta_2' = (0,1)$, $x_2' = 1$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - k$.

(iv) Player 2 deviates to $\theta_2' = (0,0)$, $x_2' = 1$: yields $U_2' = -k$.

Observe that (3) will be an equilibrium if and only if (i) through (iv) are not profitable deviations. This yields four conditions for (3) to be an equilibrium, but only conditions (i) and (ii) are binding:

Condition (i): $\alpha_1^2 < \frac{2}{3(n+1)}\alpha_2^2$ (This inequality is strict because player 1 prefers, all else equal, to not value activities.)

Condition (ii): $\alpha_1^2 > 1$ (This inequality is strict because player 1 prefers, all else equal, to not value activities.)

Condition (k): $k \leq 0$

Case 2b: Interaction, initiated by player 1 only ($x_1 = 1, x_2 = 0$)

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Observe that there will be a profitable deviation to $x_2' = 1$ if $k < 0$. There will not be a profitable deviation to $x_2' = 1$ if $k \geq 0$. Therefore, it will be sufficient, after imposing the condition $k \geq 0$, to restrict attention to deviations by player 2 in which $x_2' = 0$. There are three combinations to check.

1. $\theta_1 = (1, 0)$

   If the players follow these strategies, they receive: $U_1 = 2 \left(\frac{n+1}{n+2}\right)^2 \alpha_1^2 - 4 \frac{n+1}{(n+2)^2} \alpha_2^2 - k$, $U_2 = 2 \left(\frac{n+1}{n+2}\right)^2 \alpha_2^2 - 4 \frac{n+1}{(n+2)^2} \alpha_1^2$. In order for this to be an equilibrium, there cannot be any profitable deviations. There are seven deviations we need to check.

   (i) Player 1 deviates to $\theta_1' = (0, 1)$, $x_1' = 1$: yields $U_1' = \frac{1}{2} \left(\frac{n+1}{n+2}\right)^2 \max(\alpha_1^2, 1) - 2 \frac{n+1}{(n+2)^2} \alpha_2^2 - k$.

   (ii) Player 1 deviates to $\theta_1' = (0, 0)$, $x_1' = 1$: yields $U_1' = \frac{1}{2} \left(\frac{n+1}{n+2}\right)^2 \alpha_1^2 - 2 \frac{n+1}{(n+2)^2} \alpha_2^2 - k$.

   (iii) Player 1 deviates to $\theta_1' = (1, 0)$, $x_1' = 0$: yields $U_1' = \frac{1}{2} \left(\frac{n+1}{n+2}\right)^2 \alpha_1^2 - 2 \frac{n+1}{(n+2)^2} \alpha_2^2$.

   (iv) Player 1 deviates to $\theta_1' = (0, 1)$, $x_1' = 0$: yields $U_1' = \frac{1}{2} \left(\frac{n+1}{n+2}\right)^2$.

   (v) Player 1 deviates to $\theta_1' = (0, 0)$, $x_1' = 0$: yields $U_1' = 0$.

   (vi) Player 2 deviates to $\theta_2' = (0, 1)$, $x_2' = 0$: yields $U_2' = \frac{1}{2} \left(\frac{n+1}{n+2}\right)^2 \alpha_2^2 - 2 \frac{n+1}{(n+2)^2} \alpha_1^2$.

   (vii) Player 2 deviates to $\theta_2' = (0, 0)$, $x_2' = 0$: yields $U_2' = \frac{1}{2} \left(\frac{n+1}{n+2}\right)^2 \alpha_2^2 - 2 \frac{n+1}{(n+2)^2} \alpha_1^2$.

   Observe that (1) will be an equilibrium if and only if (i) through (vii) are not profitable deviations. This yields seven conditions for (1) to be an equilibrium, but only conditions (i), (ii), (iii), and (iv) are binding:

   Condition (i): $\alpha_1^2 \geq \frac{1}{4} + \frac{1}{n+1} \alpha_2^2$

   Condition (ii): $\alpha_1^2 > \frac{4}{3(n+1)} \alpha_2^2$ (This inequality is strict because player 1 prefers, all else equal, to not value activities.)

   Condition (iii): $k \leq \left(\frac{n+1}{n+2}\right)^2 \left[\frac{3}{2}\alpha_1^2 - \frac{2}{n+1}\alpha_2^2\right]$

   Condition (iv): $k \leq \left(\frac{n+1}{n+2}\right)^2 \left(2\alpha_1^2 - \frac{1}{2} - \frac{4}{n+1}\alpha_2^2\right)$

   Condition (k): $k \geq 0$

2. $\theta_1 = (0, 1)$

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If $\alpha_1^2 \geq 1$, there will always be a profitable deviation for player 1 to $\theta_1' = (0,0)$, $x_1' = 1$. So, assume $\alpha_1^2 < 1$. If the players follow these strategies, $U_1 = \frac{1}{2} \left(\frac{n+1}{n+2}\right)^2 - \frac{n+1}{(n+2)^2}\alpha_2^2 - k$. If player 1 deviates to $\theta_1' = (0,1)$, $x_1' = 0$, this yields: $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2$. So, there is always a profitable deviation. Thus, (2) is never an equilibrium.

3. $\theta_1 = (0,0)$

   If the players follow these strategies, $U_1 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_1^2 - \frac{n+1}{(n+2)^2}\alpha_2^2 - k$. If player 1 deviates to $\theta_1' = (1,0)$, $x_1' = 1$, this yields: $U_1' = 2\left(\frac{n+1}{n+2}\right)^2\alpha_1^2 - 2\frac{n+1}{(n+2)^2}\alpha_2^2 - k$. This is a profitable deviation if $\frac{3}{2}(n+1)\alpha_1^2 > \alpha_2^2$. Suppose this is not a profitable deviation: $\frac{3}{2}(n+1)\alpha_1^2 \leq \alpha_2^2$. If player 1 instead deviates to $\theta_1' = (0,0)$, $x_1' = 0$, this yields: $U_1' = 0$. This is a profitable deviation if $\frac{n+1}{(n+2)^2}\left(\frac{1}{2}(n+1)\alpha_1^2 - \alpha_2^2\right) < k$. Since $\frac{3}{2}(n+1)\alpha_1^2 \leq \alpha_2^2$, it follows that $\frac{1}{2}(n+1)\alpha_1^2 - \alpha_2^2 < 0$. And, since $k \geq 0$, it must be true that $\frac{n+1}{(n+2)^2}\left(\frac{1}{2}(n+1)\alpha_1^2 - \alpha_2^2\right) < k$. Hence, $\theta_1' = (0,0)$, $x_1' = 0$ must indeed be a profitable deviation. So, it follows that at least one of the two deviations will be profitable. So, no equilibrium exists with $\theta_1 = (0,0)$.

Case 2c: Interaction, initiated by player 2 only ($x_1 = 0, x_2 = 1$)

Observe that there will be a profitable deviation to $x_1' = 1$ if $k < 0$. There will not be a profitable deviation to $x_1' = 1$ if $k \geq 0$. Therefore, it will be sufficient, after imposing the condition $k \geq 0$, to restrict attention to deviations by player 1 in which $x_1' = 0$. There are three combinations to check.

1. $\theta_1 = (1,0)$

   If the players follow these strategies, they receive: $U_1 = 2\left(\frac{n+1}{n+2}\right)^2\alpha_1^2 - 4\frac{n+1}{(n+2)^2}\alpha_2^2$, $U_2 = 2\left(\frac{n+1}{n+2}\right)^2\alpha_2^2 - 4\frac{n+1}{(n+2)^2}\alpha_1^2 - k$. In order for this to be an equilibrium, there cannot be any profitable deviations. There are seven deviations we need to check.

   (i) Player 1 deviates to $\theta_1' = (0,1)$, $x_1' = 0$: yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2\max(\alpha_1^2,1) - 2\frac{n+1}{(n+2)^2}\alpha_2^2$.

   (ii) Player 1 deviates to $\theta_1' = (0,0)$, $x_1' = 0$: yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2\alpha_1^2 - 2\frac{n+1}{(n+2)^2}\alpha_2^2$.

   (iii) Player 2 deviates to $\theta_2' = (0,1)$, $x_2' = 1$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2\alpha_2^2 - 2\frac{n+1}{(n+2)^2}\alpha_1^2 - k$.

   (iv) Player 2 deviates to $\theta_2' = (0,0)$, $x_2' = 1$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2\alpha_2^2 - 2\frac{n+1}{(n+2)^2}\alpha_1^2 - k$.

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(v) Player 2 deviates to $\theta_2' = (1,0)$, $x_2' = 0$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2\alpha_2^2 - 2\frac{n+1}{(n+2)^2}\alpha_1^2$.

(vi) Player 2 deviates to $\theta_2' = (0,1)$, $x_2' = 0$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2$.

(vii) Player 2 deviates to $\theta_2' = (0,0)$, $x_2' = 0$: yields $U_2' = 0$.

Observe that (1) will be an equilibrium if and only if (i) through (vii) are not profitable deviations. This yields seven conditions for (1) to be an equilibrium, but only conditions (i), (ii), (v), and (vi) are binding:

Condition (i): $\alpha_1^2 \geq \frac{1}{4} + \frac{1}{n+1}\alpha_2^2$

Condition (ii): $\alpha_1^2 > \frac{4}{3(n+1)}\alpha_2^2$ (This inequality is strict because player 1 prefers, all else equal, to not value activities.)

Condition (v): $k \leq \left(\frac{n+1}{n+2}\right)^2\left(\frac{3}{2}\alpha_2^2 - \frac{2}{n+1}\alpha_1^2\right)$

Condition (vi): $k \leq \left(\frac{n+1}{n+2}\right)^2\left(2\alpha_2^2 - \frac{1}{2} - \frac{4}{n+1}\alpha_1^2\right)$

Condition (k): $k \geq 0$

2. $\theta_1 = (0,1)$

If $\alpha_1^2 \geq 1$, there will always be a profitable deviation for player 1 to $\theta_1' = (0,0)$, $x_1' = 0$. So, assume $\alpha_1^2 < 1$. If the players follow these strategies, $U_2 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2\alpha_2^2 - \frac{n+1}{(n+2)^2} - k$. If player 2 deviates to $\theta_2' = (1,0)$, $x_2' = 0$, this yields: $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2\alpha_2^2$, which is always profitable. So, equilibria of this type never exist.

3. $\theta_1 = (0,0)$

If players follow these strategies, $U_2 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2\alpha_1^2 - \frac{n+1}{(n+2)^2}\alpha_2^2$, $U_2 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2\alpha_2^2 - \frac{n+1}{(n+2)^2}\alpha_1^2 - k$. In order for this to be an equilibrium, there cannot be any profitable deviations. There are seven deviations we need to check.

(i) Player 1 deviates to $\theta_1' = (1,0)$, $x_1' = 1$: yields $U_1' = 2\left(\frac{n+1}{n+2}\right)^2\alpha_1^2 - 2\frac{n+1}{(n+2)^2}\alpha_2^2$.

(ii) Player 1 deviates to $\theta_1' = (0,1)$, $x_1' = 1$: yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2\max(\alpha_1^2,1) - \frac{n+1}{(n+2)^2}\alpha_2^2$.

(iii) Player 2 deviates to $\theta_2' = (0,1)$, $x_2' = 1$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - k$.

(iv) Player 2 deviates to $\theta_2' = (0,0)$, $x_2' = 1$: yields $U_2' = -k$.

(v) Player 2 deviates to $\theta_2' = (1,0)$, $x_2' = 0$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2\alpha_2^2 - \frac{n+1}{(n+2)^2}\alpha_1^2$.

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(vi) Player 2 deviates to $\theta'_2 = (0,1)$, $x'_2 = 0$: yields $U'_2 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2$.

(vii) Player 2 deviates to $\theta'_2 = (0,0)$, $x'_2 = 0$: yields $U'_2 = 0$.

Observe that (3) will be an equilibrium if and only if (i) through (vii) are not profitable deviations. This yields seven conditions for (3) to be an equilibrium, but only conditions (i), (ii), (v), and (vi) are binding:

Condition (i): $\alpha_1^2 < \frac{2}{3(n+1)}\alpha_2^2$ (This inequality is strict because player 1 prefers, all else equal, to not value activities.)

Condition (ii): $\alpha_1^2 > 1$ (This inequality is strict because player 1 prefers, all else equal, to not value activities.)

Condition (v): $k \leq 0$

If this condition is combined with condition (k), stated below, we see that we must have $k = 0$.

Condition (vi): $\left(\frac{n+1}{n+2}\right)^2 \left(\frac{1}{2}\alpha_2^2 - \frac{1}{n+1}\alpha_1^2 - \frac{1}{2}\right) \geq k$

Or, using the fact that we must have $k = 0$, this can be rewritten as: $\alpha_2^2 \geq \frac{2}{n+1}\alpha_1^2 + 1$

Condition (k): $k \geq 0$

Case 2: Combined

If we combine Cases 2a, 2b, and 2c we find the following.

1. An equilibrium with $\theta_1 = \theta_2 = (1,0)$ and interaction between the players exists if:

   (i) $\alpha_1^2 \geq \frac{1}{4} + \frac{1}{n+1}\alpha_2^2$

   (ii) $\alpha_1^2 > \frac{4}{3(n+1)}\alpha_2^2$

   (iii) $k \leq 0$ or $k \geq 0$, $k \leq \left(\frac{n+1}{n+2}\right)^2 \left(\frac{3}{2}\alpha_2^2 - \frac{2}{n+1}\alpha_1^2\right)$, and $k \leq \left(\frac{n+1}{n+2}\right)^2 \left(2\alpha_2^2 - \frac{1}{2} - \frac{4}{n+1}\alpha_1^2\right)$.

   Since $\left(\frac{n+1}{n+2}\right)^2 \left(\frac{3}{2}\alpha_2^2 - \frac{2}{n+1}\alpha_1^2\right) \geq 0$ and $\left(\frac{n+1}{n+2}\right)^2 \left(2\alpha_2^2 - \frac{1}{2} - \frac{4}{n+1}\alpha_1^2\right) \geq 0$, condition (iii) can be restated as: $k \leq \left(\frac{n+1}{n+2}\right)^2 \min\left(\frac{3}{2}\alpha_2^2 - \frac{2}{n+1}\alpha_1^2, 2\alpha_2^2 - \frac{1}{2} - \frac{4}{n+1}\alpha_1^2\right) = \overline{k}_3$.

2. An equilibrium with $\theta_1 = (0,1)$, $\theta_2 = (1,0)$ and interaction between the players exists if:

(i) $\alpha_1^2 \leq \frac{1}{4} + \frac{1}{2(n+1)}\alpha_2^2$

(ii) $\alpha_1^2 < 1$

(iii) $k \leq 0$

3. An equilibrium with $\theta_1 = (0,0)$, $\theta_2 = (1,0)$ and interaction between the players exists if:

(i) $1 < \alpha_1^2 < \frac{2}{3(n+1)}\alpha_2^2$

(ii) $k \leq 0$

Additional combinations to consider:

In order to complete the proof, we also need to show that no equilibria exist in which player 2 is not a scholar. There are twelve cases we need to consider.

1. $\theta_1 = (1,0)$, $\theta_2 = (0,1)$ and no interaction ($x_1 = x_2 = 0$)

   If both players follow these strategies, $U_2 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2$. If player 2 deviates to $\theta_2' = (1,0)$, $x_2' = 0$, this yields $\frac{1}{2}\left(\frac{n+1}{n+2}\right)^2\alpha_2^2 - \frac{n+1}{(n+2)^2}\alpha_1^2$. This deviation is profitable if $1 < \alpha_2^2 - \frac{2}{n+1}\alpha_1^2$. Since $\alpha_2 \geq \alpha_1$, a sufficient condition for this deviation to be profitable is $1 < \alpha_2^2 - \frac{2}{n+1}\alpha_2^2$, or $\frac{n-1}{n+1}\alpha_2^2 > 1$, which is true by assumption.

2. $\theta_1 = (0,1)$, $\theta_2 = (0,1)$ and no interaction ($x_1 = x_2 = 0$)

   If both players follow these strategies, $U_2 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - \frac{n+1}{(n+2)^2}$. If player 2 deviates to $\theta_2' = (1,0)$, $x_2' = 0$, this yields $\frac{1}{2}\left(\frac{n+1}{n+2}\right)^2\alpha_2^2$. This deviation is profitable if $\alpha_2^2 > \frac{n-1}{n+1}$, which is true by assumption.

3. $\theta_1 = (0,0)$, $\theta_2 = (0,1)$ and no interaction ($x_1 = x_2 = 0$)

   If both players follow these strategies, $U_2 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2$. If player 2 deviates to $\theta_2' = (1,0)$, $x_2' = 0$, this yields $\frac{1}{2}\left(\frac{n+1}{n+2}\right)^2\alpha_2^2$, which is always profitable.

4. $\theta_1 = (1,0)$, $\theta_2 = (0,0)$ and no interaction ($x_1 = x_2 = 0$)

   If both players follow these strategies, $U_2 = 0$. If player 2 deviates to $\theta_2' = (0,1)$, $x_2' = 0$, this yields $\frac{1}{2}\left(\frac{n+1}{n+2}\right)^2$, which is always profitable.

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5. $\theta_1 = (0,1)$, $\theta_2 = (0,0)$ and no interaction ($x_1 = x_2 = 0$)

If both players follow these strategies, $U_2 = 0$. If player 2 deviates to $\theta_2' = (1,0)$, $x_2' = 0$, this yields $\frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_2^2$, which is always profitable.

6. $\theta_1 = (0,0)$, $\theta_2 = (0,0)$ and no interaction ($x_1 = x_2 = 0$)

If both players follow these strategies, $U_2 = 0$. If player 2 deviates to $\theta_2' = (1,0)$, $x_2' = 0$, this yields $\frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_2^2$, which is always profitable.

7. $\theta_1 = (1,0)$, $\theta_2 = (0,1)$ and interaction ($x_1 = 1$ or $x_2 = 1$)

If $\alpha_1^2 \geq 1$. Then, if both players follow these strategies, $U_2 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_2^2 - \frac{n+1}{(n+2)^2}\alpha_1^2 - kx_2$. If player 2 deviates to $\theta_2' = (1,0)$, $x_2' = x_2$, this yields $U_2' = 2\left(\frac{n+1}{n+2}\right)^2 \alpha_2^2 - 2\frac{n+1}{(n+2)^2}\alpha_1^2 - kx_2$. This deviation is profitable if $\alpha_2^2 > \frac{2}{3(n+1)}\alpha_1^2$. Since $\alpha_2 \geq \alpha_1$ by assumption, this will indeed be a profitable deviation. Now suppose $\alpha_1^2 < 1$. Then, it is profitable for player 1 to deviate to $\theta_1' = (0,0)$, $x_1' = x_1$. So, a profitable deviation always exists.

8. $\theta_1 = (0,1)$, $\theta_2 = (0,1)$ and interaction ($x_1 = 1$ or $x_2 = 1$)

If both players follow these strategies, $U_2 = 2\left(\frac{n+1}{n+2}\right)^2 - 4\frac{n+1}{(n+2)^2} - kx_2$. If player 2 deviates to $\theta_2' = (1,0)$, $x_2' = x_2$, this yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_2^2 - 2\frac{n+1}{(n+2)^2} - kx_2$. This deviation is profitable if $\alpha_2^2 > \frac{4n}{n+1}$, which is true by assumption.

9. $\theta_1 = (0,0)$, $\theta_2 = (0,1)$ and interaction ($x_1 = 1$ or $x_2 = 1$)

If both players follow these strategies, $U_1 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - \frac{n+1}{(n+2)^2} - kx_1$. If player 1 deviates to $\theta_1' = (0,1)$, $x_1' = x_1$, this yields $U_1' = 2\left(\frac{n+1}{n+2}\right)^2 - 2\frac{n+1}{(n+2)^2} - kx_1$. This deviation is profitable if $\frac{3}{2} > \frac{1}{n+1}$, which is always true.

10. $\theta_1 = (1,0)$, $\theta_2 = (0,0)$ and interaction ($x_1 = 1$ or $x_2 = 1$)

If both players follow these strategies, $U_2 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_2^2 - \frac{n+1}{(n+2)^2}\alpha_1^2 - kx_2$. If player 2 deviates to $\theta_2' = (1,0)$, $x_2' = x_2$, this yields $U_2' = 2\left(\frac{n+1}{n+2}\right)^2 \alpha_2^2 - 2\frac{n+1}{(n+2)^2}\alpha_1^2 - kx_2$. This deviation is profitable if $\alpha_2^2 > \frac{2}{3(n+1)}\alpha_1^2$. Since $\alpha_2 \geq \alpha_1$ by assumption, this will indeed be a profitable deviation.

11. $\theta_1 = (0,1)$, $\theta_2 = (0,0)$ and interaction ($x_1 = 1$ or $x_2 = 1$)

If both players follow these strategies, $U_2 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - \frac{n+1}{(n+2)^2} - kx_2$. If player 2 deviates to $\theta_2' = (1, 0)$, $x_2' = x_2$, this yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_2^2 - \frac{n+1}{(n+2)^2} - kx_2$. This deviation is profitable if $\alpha_2^2 > 1$, which is true by assumption.

12. $\theta_1 = (0, 0)$, $\theta_2 = (0, 0)$ and interaction ($x_1 = 1$ or $x_2 = 1$)

   If both players follow these strategies, $U_2 = -kx_2$. If player 2 deviates to $\theta_2' = (1, 0)$, $x_2' = x_2$, this yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_2^2 - kx_2$. This deviation is always profitable.

This establishes that no equilibria exist in which player 2 is not a scholar. This completes the proof. $\qquad\square$

Proof of Proposition 3. Suppose $\alpha_2 \geq \alpha_1$ and $\alpha_1^2 < \overline{\alpha}_L$. We will characterize the equilibria that exist as a function of α_1, α_2, k, and n.

It will be useful, in characterizing the equilibria, to use the same shorthand as we used in the proof of Proposition 1. Once again, we will examine all combinations of the following form to see, under what parameters, they are equilibria of the game: $\{((\theta_1, x_1), (\theta_2, x_2)) : \theta_i \in \{(1, 0), (0, 1), (0, 0)\}, x_i \in \{0, 1\}\}$. We will first examine combinations in which player 1 is a musician ($\theta_1 = (0, 1)$). Later on, we will show that no equilibria exist in which player 1 is not a musician.

Case 1: No interaction ($x_1 = x_2 = 0$)

Since we assume player 1 is a musician ($\theta_1 = (0, 1)$), we will denote combinations by player 2's choice of θ_2. There are three combinations to check.

1. $\theta_2 = (0, 1)$

   If the players follow these strategies, they receive: $U_1 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - \frac{n+1}{(n+2)^2}$, $U_2 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - \frac{n+1}{(n+2)^2}$. In order for this to be an equilibrium, there cannot be any profitable deviations. There are ten deviations we need to check.

   (i) Player 1 deviates to $\theta_1' = (1, 0)$, $x_1' = 0$: yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_1^2$.

   (ii) Player 1 deviates to $\theta_1' = (0, 0)$, $x_1' = 0$: yields $U_1' = 0$.

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(iii) Player 1 deviates to $\theta_1' = (0,1)$, $x_1' = 1$: yields $U_1' = 2\left(\frac{n+1}{n+2}\right)^2 - 2\frac{n+1}{(n+2)^2} - k$.

(iv) Player 1 deviates to $\theta_1' = (1,0)$, $x_1' = 1$: yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - \frac{n+1}{(n+2)^2} - k$.

(v) Player 1 deviates to $\theta_1' = (0,0)$, $x_1' = 1$: yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - \frac{n+1}{(n+2)^2} - k$.

(vi) Player 2 deviates to $\theta_2' = (1,0)$, $x_2' = 0$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_2^2$.

(vii) Player 2 deviates to $\theta_2' = (0,0)$, $x_2' = 0$: yields $U_2' = 0$.

(viii) Player 2 deviates to $\theta_2' = (0,1)$, $x_2' = 1$: yields $U_2' = 2\left(\frac{n+1}{n+2}\right)^2 - 2\frac{n+1}{(n+2)^2} - k$.

(ix) Player 2 deviates to $\theta_2' = (1,0)$, $x_2' = 1$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \max(\alpha_2^2, 1) - \frac{n+1}{(n+2)^2} - k$.

(x) Player 2 deviates to $\theta_2' = (0,0)$, $x_2' = 1$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - \frac{n+1}{(n+2)^2} - k$.

Observe that (1) will be an equilibrium if and only if (i) through (x) are not profitable deviations. This yields ten conditions for (1) to be an equilibrium, but only conditions (iii), (vi), and (ix) are binding:

Condition (iii): $k \geq \left(\frac{n+1}{n+2}\right)^2 \left(\frac{3}{2} - \frac{1}{n+1}\right)$

Condition (vi): $\frac{n-1}{n+1} \geq \alpha_2^2$

Condition (ix): $k \geq \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 (\alpha_2^2 - 1)$

We can combine conditions (iii) and (ix) as follows:

$k \geq \left(\frac{n+1}{n+2}\right)^2 \max\left(\frac{3}{2} - \frac{1}{n+1}, \frac{1}{2}\alpha_2^2 - \frac{1}{2}\right) = \overline{k}_1$

2. $\theta_2 = (1,0)$

If the players follow these strategies, they receive: $U_1 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2$, $U_2 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_2^2$. In order for this to be an equilibrium, there cannot be any profitable deviations. There are ten deviations we need to check.

(i) Player 2 deviates to $\theta_2' = (0,1)$, $x_2' = 0$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - \frac{n+1}{(n+2)^2}$.

(ii) Player 2 deviates to $\theta_2' = (0,0)$, $x_2' = 0$: yields $U_2' = 0$.

(iii) Player 2 deviates to $\theta_2' = (1,0)$, $x_2' = 1$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \max(1, \alpha_2^2) - \frac{n+1}{(n+2)^2} - k$.

(iv) Player 2 deviates to $\theta_2' = (0,1)$, $x_2' = 1$: yields $U_2' = 2\left(\frac{n+1}{n+2}\right)^2 - 2\frac{n+1}{(n+2)^2} - k$.

(v) Player 2 deviates to $\theta_2' = (0,0)$, $x_2' = 1$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - \frac{n+1}{(n+2)^2} - k$.

(vi) Player 1 deviates to $\theta_1' = (1,0)$, $x_1' = 0$: yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_1^2 - \frac{n+1}{(n+2)^2}\alpha_2^2$.

(vii) Player 1 deviates to $\theta_1' = (0,0)$, $x_1' = 0$: yields $U_1' = 0$.

(viii) Player 1 deviates to $\theta_1' = (1,0)$, $x_1' = 1$: yields $U_1' = 2\left(\frac{n+1}{n+2}\right)^2 \alpha_1^2 - 2\frac{n+1}{(n+2)^2}\alpha_2^2 - k$.

(ix) Player 1 deviates to $\theta_1' = (0,1)$, $x_1' = 1$: yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - \frac{n+1}{(n+2)^2}\alpha_2^2 - k$.

(x) Player 1 deviates to $\theta_1' = (0,0)$, $x_1' = 1$: yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_1^2 - \frac{n+1}{(n+2)^2}\alpha_2^2 - k$.

Observe that (2) will be an equilibrium if and only if (i) through (x) are not profitable deviations. This yields ten conditions for (2) to be an equilibrium, but only conditions (i), (iii), (iv), and (viii) are binding:

Condition (i): $\alpha_2^2 \geq \frac{n-1}{n+1}$

Condition (iii): $k \geq \left(\frac{n+1}{n+2}\right)^2\left(-\frac{1}{n+1}\right)$

Condition (iv): $k \geq \left(\frac{n+1}{n+2}\right)^2\left(\frac{2n}{n+1} - \frac{1}{2}\alpha_2^2\right)$

Condition (viii): $k \geq \left(\frac{n+1}{n+2}\right)^2\left(2\alpha_1^2 - \frac{1}{2} - \frac{2\alpha_2^2}{n+1}\right)$

We can combine conditions (iii), (iv), and (viii) as follows:

$$k \geq \left(\frac{n+1}{n+2}\right)^2 \max\left(-\frac{1}{n+1}, \frac{2n}{n+1} - \frac{1}{2}\alpha_2^2, 2\alpha_1^2 - \frac{1}{2} - \frac{2\alpha_2^2}{n+1}\right) = \bar{k}_2$$

3. $\theta_2 = (0,0)$

If the players follow these strategies, $U_2 = 0$. If player 2 deviates to $\theta_2' = (1,0)$, $x_2' = 0$, this yields: $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2\alpha_2^2$. So, there is always a profitable deviation. Thus, (3) is never an equilibrium.

Case 2a: Interaction, initiated by both players ($x_1 = x_2 = 1$)

Observe that there will be a profitable deviation to $x_i' = 0$ if $k > 0$. If $k \leq 0$, deviating to $x_i' = 0$ is (weakly) unprofitable. Therefore, it will be sufficient, after imposing the condition $k \leq 0$, to restrict attention to deviations in which $x_i' = 1$. There are three combinations to check.

1. $\theta_2 = (0,1)$

If the players follow these strategies, they receive: $U_1 = 2 \left(\frac{n+1}{n+2}\right)^2 - 4\frac{n+1}{(n+2)^2} - k$, $U_2 = 2 \left(\frac{n+1}{n+2}\right)^2 - 4\frac{n+1}{(n+2)^2} - k$. In order for this to be an equilibrium, there cannot be any profitable deviations. There are four deviations we need to check.

(i) Player 1 deviates to $\theta_1' = (1,0)$, $x_1' = 1$: yields $U_1' = \frac{1}{2} \left(\frac{n+1}{n+2}\right)^2 - 2\frac{n+1}{(n+2)^2} - k$.

(ii) Player 1 deviates to $\theta_1' = (0,0)$, $x_1' = 1$: yields $U_1' = \frac{1}{2} \left(\frac{n+1}{n+2}\right)^2 - 2\frac{n+1}{(n+2)^2} - k$.

(iii) Player 2 deviates to $\theta_2' = (1,0)$, $x_2' = 1$: yields $U_2' = \frac{1}{2} \left(\frac{n+1}{n+2}\right)^2 \max(\alpha_2^2, 1) - 2\frac{n+1}{(n+2)^2} - k$.

(iv) Player 2 deviates to $\theta_2' = (0,0)$, $x_2' = 1$: yields $U_2' = \frac{1}{2} \left(\frac{n+1}{n+2}\right)^2 - 2\frac{n+1}{(n+2)^2} - k$.

Observe that (1) will be an equilibrium if and only if (i) through (iv) are not profitable deviations. Only condition (iii) is binding:

Condition (iii): $\alpha_2^2 \le \frac{4n}{n+1}$

Condition (k): $k \le 0$

2. $\theta_2 = (1,0)$

If $\alpha_2^2 < 1$, there will always be a profitable deviation for player 2 to $\theta_2' = (0,0)$, $x_2' = 1$. So we assume $\alpha_2^2 \ge 1$. If the players follow these strategies, they receive: $U_1 = \frac{1}{2} \left(\frac{n+1}{n+2}\right)^2 - \frac{n+1}{(n+2)^2}\alpha_2^2 - k$, $U_2 = \frac{1}{2} \left(\frac{n+1}{n+2}\right)^2 \alpha_2^2 - \frac{n+1}{(n+2)^2} - k$. In order for this to be an equilibrium, there cannot be any profitable deviations. There are four deviations we need to check.

(i) Player 1 deviates to $\theta_1' = (1,0)$, $x_1' = 1$: yields $U_1' = 2 \left(\frac{n+1}{n+2}\right)^2 \alpha_1^2 - 2\frac{n+1}{(n+2)^2}\alpha_2^2 - k$.

(ii) Player 1 deviates to $\theta_1' = (0,0)$, $x_1' = 1$: yields $U_1' = \frac{1}{2} \left(\frac{n+1}{n+2}\right)^2 \alpha_1^2 - \frac{n+1}{(n+2)^2}\alpha_2^2 - k$.

(iii) Player 2 deviates to $\theta_2' = (0,1)$, $x_2' = 1$: yields $U_2' = 2 \left(\frac{n+1}{n+2}\right)^2 - 2\frac{n+1}{(n+2)^2} - k$.

(iv) Player 2 deviates to $\theta_2' = (0,0)$, $x_2' = 1$: yields $U_2' = \frac{1}{2} \left(\frac{n+1}{n+2}\right)^2 - \frac{n+1}{(n+2)^2} - k$.

Observe that (2) will be an equilibrium if and only if (i) through (iv) are not profitable deviations. This yields four conditions for (2) to be an equilibrium, but only condition (iii) is binding:

Condition (iii): $\alpha_2^2 \ge 4 - \frac{2}{n+1}$

Condition (k): $k \le 0$

3. $\theta_2 = (0,0)$

If the players follow these strategies, they receive: $U_1 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - \frac{n+1}{(n+2)^2} - k$, $U_2 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - \frac{n+1}{(n+2)^2} - k$. If player 2 deviates to $\theta_2' = (0,1)$, $x_2' = 1$, this yields $U_2' = 2\left(\frac{n+1}{n+2}\right)^2 - 2\frac{n+1}{(n+2)^2} - k$. This deviation is always profitable. So, this is never an equilibrium.

Case 2b: Interaction, initiated by player 2 only ($x_1 = 0, x_2 = 1$)

Observe that there will be a profitable deviation to $x_1' = 1$ if $k < 0$. There will not be a profitable deviation to $x_1' = 1$ if $k \geq 0$. Therefore, it will be sufficient, after imposing the condition $k \geq 0$, to restrict attention to deviations by player 1 in which $x_1' = 0$. There are three combinations to check.

1. $\theta_2 = (0,1)$

If the players follow these strategies, they receive: $U_1 = 2\left(\frac{n+1}{n+2}\right)^2 - 4\frac{n+1}{(n+2)^2}$, $U_2 = 2\left(\frac{n+1}{n+2}\right)^2 - 4\frac{n+1}{(n+2)^2} - k$. In order for this to be an equilibrium, there cannot be any profitable deviations. There are seven deviations we need to check.

   (i) Player 1 deviates to $\theta_1' = (1,0)$, $x_1' = 0$: yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - 2\frac{n+1}{(n+2)^2}$.

   (ii) Player 1 deviates to $\theta_1' = (0,0)$, $x_1' = 0$: yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - 2\frac{n+1}{(n+2)^2}$.

   (iii) Player 2 deviates to $\theta_2' = (1,0)$, $x_2' = 1$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \max(\alpha_2^2, 1) - 2\frac{n+1}{(n+2)^2} - k$.

   (iv) Player 2 deviates to $\theta_2' = (0,0)$, $x_2' = 1$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - 2\frac{n+1}{(n+2)^2} - k$.

   (v) Player 2 deviates to $\theta_2' = (0,1)$, $x_2' = 0$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - 2\frac{n+1}{(n+2)^2}$.

   (vi) Player 2 deviates to $\theta_2' = (1,0)$, $x_2' = 0$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_2^2$.

   (vii) Player 2 deviates to $\theta_2' = (0,0)$, $x_2' = 0$: yields $U_2' = 0$.

Observe that (1) will be an equilibrium if and only if (i) through (vii) are not profitable deviations. This yields seven conditions for (1) to be an equilibrium, but only conditions (iii), (v), and (vi) are binding:

Condition (iii): $\frac{4n}{n+1} \geq \alpha_2^2$

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Condition (v): $\left(\frac{n+1}{n+2}\right)^2 \left(\frac{3}{2} - \frac{2}{n+1}\right) \geq k$

Condition (vi): $\left(\frac{n+1}{n+2}\right)^2 \left(2\left(\frac{n-1}{n+1}\right) - \frac{1}{2}\alpha_2^2\right) \geq k$

Condition (k): $k \geq 0$

2. $\theta_2 = (1, 0)$

If $\alpha_2^2 < 1$, there will always be a profitable deviation for player 2 to $\theta_2' = (0,0)$, $x_2' = 1$. So, assume $\alpha_2^2 \geq 1$. If the players follow these strategies, $U_2 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_2^2 - \frac{n+1}{(n+2)^2} - k$. If player 2 deviates to $\theta_2' = (1, 0)$, $x_2' = 0$, this yields: $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_2^2$. So, there is always a profitable deviation. Thus, (2) is never an equilibrium.

3. $\theta_2 = (0, 0)$

If the players follow these strategies, $U_2 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - \frac{n+1}{(n+2)^2} - k$. If player 2 deviates to $\theta_2' = (0, 1)$, $x_2' = 1$, this yields: $U_2' = 2\left(\frac{n+1}{n+2}\right)^2 - 2\frac{n+1}{(n+2)^2} - k$. This is always a profitable deviation. So, no equilibrium exists with $\theta_2 = (0, 0)$.

Case 2c: Interaction, initiated by player 1 only ($x_1 = 1, x_2 = 0$)

Observe that there will be a profitable deviation to $x_2' = 1$ if $k < 0$. There will not be a profitable deviation to $x_2' = 1$ if $k \geq 0$. Therefore, it will be sufficient, after imposing the condition $k \geq 0$, to restrict attention to deviations by player 2 in which $x_2' = 0$. There are three combinations to check.

1. $\theta_2 = (0, 1)$

If the players follow these strategies, they receive: $U_1 = 2\left(\frac{n+1}{n+2}\right)^2 - 4\frac{n+1}{(n+2)^2} - k$, $U_2 = 2\left(\frac{n+1}{n+2}\right)^2 - 4\frac{n+1}{(n+2)^2}$. In order for this to be an equilibrium, there cannot be any profitable deviations. There are seven deviations we need to check.

(i) Player 1 deviates to $\theta_1' = (1, 0)$, $x_1' = 1$: yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - 2\frac{n+1}{(n+2)^2} - k$.

(ii) Player 1 deviates to $\theta_1' = (0, 0)$, $x_1' = 1$: yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - 2\frac{n+1}{(n+2)^2} - k$.

(iii) Player 1 deviates to $\theta_1' = (0, 1)$, $x_1' = 0$: yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - 2\frac{n+1}{(n+2)^2}$.

(iv) Player 1 deviates to $\theta_1' = (1, 0)$, $x_1' = 0$: yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_1^2$.

(v) Player 1 deviates to $\theta_1' = (0, 0)$, $x_1' = 0$: yields $U_1' = 0$.

(vi) Player 2 deviates to $\theta_2' = (1,0)$, $x_2' = 0$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \max(\alpha_2^2, 1) - 2\frac{n+1}{(n+2)^2}$.

(vii) Player 2 deviates to $\theta_2' = (0,0)$, $x_2' = 0$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - 2\frac{n+1}{(n+2)^2}$.

Observe that (1) will be an equilibrium if and only if (i) through (vii) are not profitable deviations. This yields seven conditions for (1) to be an equilibrium, but only conditions (iii), (iv), and (vi) are binding:

Condition (iii): $k \leq \left(\frac{n+1}{n+2}\right)^2 \left(\frac{3}{2} - \frac{2}{n+1}\right)$

Condition (iv): $k \leq \left(\frac{n+1}{n+2}\right)^2 \left(2\left(\frac{n-1}{n+1}\right) - \frac{1}{2}\alpha_1^2\right)$

Condition (vi): $\frac{4n}{n+1} \geq \alpha_2^2$

Condition (k): $k \geq 0$

2. $\theta_2 = (1,0)$

If $\alpha_2^2 < 1$, there will always be a profitable deviation for player 2 to $\theta_2' = (0,0)$, $x_2' = 0$. So, assume $\alpha_2^2 \geq 1$. If the players follow these strategies, $U_1 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - \frac{n+1}{(n+2)^2}\alpha_2^2 - k$. If player 1 deviates to $\theta_1' = (0,1)$, $x_1' = 0$, this yields: $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2$, which is always profitable. So, equilibria of this type never exist.

3. $\theta_2 = (0,0)$

If players follow these strategies, $U_2 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - \frac{n+1}{(n+2)^2}$. If player 2 deviates to $\theta_2' = (0,1)$, $x_2' = 0$, this yields: $U_2' = 2\left(\frac{n+1}{n+2}\right)^2 - 2\frac{n+1}{(n+2)^2}$, which is always profitable. So, equilibria of this type never exist.

Case 2: Combined

If we combine Cases 2a, 2b, and 2c we find the following.

1. An equilibrium with $\theta_1 = \theta_2 = (0,1)$ and interaction between the players exists if:

(i) $\alpha_2^2 \leq \frac{4n}{n+1}$

(ii) $k \leq \left(\frac{n+1}{n+2}\right)^2 \min\left(\frac{3}{2} - \frac{2}{n+1}, 2\left(\frac{n-1}{n+1}\right) - \frac{1}{2}\alpha_1^2\right)$

2. An equilibrium with $\theta_1 = (0,1)$, $\theta_2 = (1,0)$ and interaction between the players exists if:

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(i) $\alpha_2^2 \geq 4 - \frac{2}{n+1}$

(ii) $k \leq 0$

Additional combinations to consider:

In order to complete the proof, we also need to show that no equilibria exist in which player 1 is not a musician. There are twelve cases we need to consider.

1. $\theta_1 = (1,0)$, $\theta_2 = (1,0)$ and no interaction ($x_1 = x_2 = 0$)

   If both players follow these strategies, $U_1 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_1^2 - \frac{n+1}{(n+2)^2}\alpha_2^2$. If player 1 deviates to $\theta_1' = (0,1)$, $x_1' = 0$, this yields $\frac{1}{2}\left(\frac{n+1}{n+2}\right)^2$. This deviation is profitable if $\frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_1^2 - \frac{n+1}{(n+2)^2}\alpha_2^2 < \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2$. Since $\alpha_1 \leq \alpha_2$, a sufficient condition for this deviation to be profitable is $\frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_1^2 - \frac{n+1}{(n+2)^2}\alpha_1^2 < \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2$, or $(n-1)\alpha_1^2 < n+1$. This holds by assumption.

2. $\theta_1 = (1,0)$, $\theta_2 = (0,1)$ and no interaction ($x_1 = x_2 = 0$)

   If both players follow these strategies, $U_1 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_1^2$ and $U_2 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2$. If player 1 deviates to $\theta_1' = (0,1)$, $x_1' = 0$, this yields $\frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - \frac{n+1}{(n+2)^2}$. This deviation is profitable if $\frac{n-1}{n+1} > \alpha_1^2$, which is true by assumption.

3. $\theta_1 = (1,0)$, $\theta_2 = (0,0)$ and no interaction ($x_1 = x_2 = 0$)

   If both players follow these strategies, $U_2 = 0$. If player 2 deviates to $\theta_2' = (0,1)$, $x_2' = 0$, this yields $\frac{1}{2}\left(\frac{n+1}{n+2}\right)^2$, which is always profitable.

4. $\theta_1 = (0,0)$, $\theta_2 = (1,0)$ and no interaction ($x_1 = x_2 = 0$)

   If both players follow these strategies, $U_1 = 0$. If player 1 deviates to $\theta_1' = (0,1)$, $x_1' = 0$, this yields $\frac{1}{2}\left(\frac{n+1}{n+2}\right)^2$, which is always profitable.

5. $\theta_1 = (0,0)$, $\theta_2 = (0,1)$ and no interaction ($x_1 = x_2 = 0$)

   If both players follow these strategies, $U_1 = 0$. If player 1 deviates to $\theta_1' = (1,0)$, $x_1' = 0$, this yields $\frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_1^2$, which is always profitable.

6. $\theta_1 = (0,0)$, $\theta_2 = (0,0)$ and no interaction ($x_1 = x_2 = 0$)

   If both players follow these strategies, $U_1 = 0$. If player 1 deviates to $\theta_1' = (0,1)$, $x_1' = 0$, this yields $\frac{1}{2}\left(\frac{n+1}{n+2}\right)^2$, which is always profitable.

7. $\theta_1 = (1,0)$, $\theta_2 = (1,0)$ and interaction ($x_1 = 1$ or $x_2 = 1$)

If both players follow these strategies, $U_1 = 2\left(\frac{n+1}{n+2}\right)^2 \alpha_1^2 - 4\frac{n+1}{(n+2)^2}\alpha_2^2 - kx_1$. If player 1 deviates to $\theta_1' = (0,1)$, $x_1' = x_1$, this yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - 2\frac{n+1}{(n+2)^2}\alpha_2^2 - kx_1$. This deviation is profitable if $1 > 4\alpha_1^2 - \frac{4}{n+1}\alpha_2^2$. Since $\alpha_2 \geq \alpha_1$, a sufficient condition for the deviation to be profitable is that $1 > 4\alpha_1^2 - \frac{4}{n+1}\alpha_1^2$, or $4n * \alpha_1^2 < n + 1$. This is true by assumption.

8. $\theta_1 = (1,0)$, $\theta_2 = (0,1)$ and interaction ($x_1 = 1$ or $x_2 = 1$)

If both players follow these strategies, $U_1 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - \frac{n+1}{(n+2)^2} - kx_1$. If player 1 deviates to $\theta_1' = (0,1)$, $x_1' = x_1$, this yields $U_1' = 2\left(\frac{n+1}{n+2}\right)^2 - 2\frac{n+1}{(n+2)^2} - kx_1$. This deviation is always profitable.

9. $\theta_1 = (1,0)$, $\theta_2 = (0,0)$ and interaction ($x_1 = 1$ or $x_2 = 1$)

If both players follow these strategies, $U_1 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_1^2 - \frac{n+1}{(n+2)^2}\alpha_2^2 - kx_1$. If player 1 deviates to $\theta_1' = (0,1)$, $x_1' = x_1$, this yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - kx_1$. This deviation is profitable if $1 > \alpha_1^2 - \frac{2}{n+1}\alpha_2^2$. Since $\alpha_2^2 \geq \alpha_1^2$, a sufficient condition for the deviation to be profitable is $1 > \alpha_1^2 - \frac{2}{n+1}\alpha_1^2$, or $1 > \frac{n-1}{n+1}\alpha_1^2$. This is true by assumption.

10. $\theta_1 = (0,0)$, $\theta_2 = (1,0)$ and interaction ($x_1 = 1$ or $x_2 = 1$)

If both players follow these strategies, $U_1 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_1^2 - \frac{n+1}{(n+2)^2}\alpha_2^2 - kx_1$. If player 1 deviates to $\theta_1' = (0,1)$, $x_1' = x_1$, this yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - \frac{n+1}{(n+2)^2}\alpha_1^2 - kx_1$. This deviation is profitable if $\alpha_1^2 < 1$, which is true by assumption.

11. $\theta_1 = (0,0)$, $\theta_2 = (0,1)$ and interaction ($x_1 = 1$ or $x_2 = 1$)

If both players follow these strategies, $U_1 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - \frac{n+1}{(n+2)^2} - kx_1$. If player 1 deviates to $\theta_2' = (0,1)$, $x_1' = x_1$, this yields $U_1' = 2\left(\frac{n+1}{n+2}\right)^2 - 2\frac{n+1}{(n+2)^2} - kx_1$. This deviation is profitable if $\frac{3}{2} > \frac{1}{n+1}$, which is always true.

12. $\theta_1 = (0,0)$, $\theta_2 = (0,0)$ and interaction ($x_1 = 1$ or $x_2 = 1$)

If both players follow these strategies, $U_1 = -kx_1$. If player 1 deviates to $\theta_1' = (0,1)$, $x_1' = x_1$, this yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - kx_1$. This deviation is always profitable.

This establishes that no equilibria exist in which player 1 is not a musician. This completes the proof. □

Proof of Proposition 4. Suppose $\alpha_2 \geq \alpha_1$, and $\overline{\alpha}_H \geq \alpha_1, \alpha_2 \geq \overline{\alpha}_L$. We will characterize the equilibria that exist as a function of α_1, α_2, k, and n.

It will be useful, in characterizing the equilibria, to use the same shorthand as we used in the proof of Proposition 1. Once again, we will examine all combinations of the following form to see, under what parameters, they are equilibria of the game: $\{((\theta_1, x_1), (\theta_2, x_2)) : \theta_i \in \{(1,0), (0,1), (0,0)\}, x_i \in \{0,1\}\}$.

Case 1: No interaction ($x_1 = x_2 = 0$)

There are nine types of combinations to check. We will consider each in turn.

1. $\theta_1 = \theta_1 = (1,0)$

   If the players follow these strategies, they receive: $U_1 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_1^2 - \frac{n+1}{(n+2)^2}\alpha_2^2$, $U_2 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_2^2 - \frac{n+1}{(n+2)^2}\alpha_1^2$. In order for this to be an equilibrium, there cannot be any profitable deviations. There are ten deviations we need to check.

   (i) Player 1 deviates to $\theta_1' = (0,1)$, $x_1' = 0$: yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2$.

   (ii) Player 1 deviates to $\theta_1' = (0,0)$, $x_1' = 0$: yields $U_1' = 0$.

   (iii) Player 1 deviates to $\theta_1' = (1,0)$, $x_1' = 1$: yields $U_1' = 2\left(\frac{n+1}{n+2}\right)^2 \alpha_1^2 - 2\frac{n+1}{(n+2)^2}\alpha_2^2 - k$.

   (iv) Player 1 deviates to $\theta_1' = (0,1)$, $x_1' = 1$: yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \max(\alpha_1^2, 1) - \frac{n+1}{(n+2)^2}\alpha_2^2 - k$.

   (v) Player 1 deviates to $\theta_1' = (0,0)$ and $x_1' = 1$: yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_1^2 - \frac{n+1}{(n+2)^2}\alpha_2^2 - k$.

   (vi) Player 2 deviates to $\theta_2' = (0,1)$, $x_2' = 0$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2$.

   (vii) Player 2 deviates to $\theta_2' = (0,0)$, $x_2' = 0$: yields $U_2' = 0$.

   (viii) Player 2 deviates to $\theta_2' = (1,0)$, $x_2' = 1$: yields $U_2' = 2\left(\frac{n+1}{n+2}\right)^2 \alpha_2^2 - 2\frac{n+1}{(n+2)^2}\alpha_1^2 - k$.

   (ix) Player 2 deviates to $\theta_2' = (0,1)$, $x_2' = 1$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \max(\alpha_2^2, 1) - \frac{n+1}{(n+2)^2}\alpha_1^2 - k$.

   (x) Player 2 deviates to $\theta_2' = (0,0)$ and $x_2' = 1$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_2^2 - \frac{n+1}{(n+2)^2}\alpha_1^2 - k$.

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Observe that (1) will be an equilibrium if and only if (i) through (x) are not profitable deviations. This yields ten conditions for (1) to be an equilibrium, but only conditions (i), (iv), and (viii) are binding:

Condition (i): $\alpha_1^2 \geq 1 + \frac{2}{n+1}\alpha_2^2$

Condition (iv): $k \geq \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 (1 - \alpha_1^2)$

Condition (viii): $k \geq \left(\frac{n+1}{n+2}\right)^2 \left(\frac{3}{2}\alpha_2^2 - \frac{1}{n+1}\alpha_1^2\right)$

If we combine conditions (iv) and (viii), we obtain:

$$k \geq \left(\frac{n+1}{n+2}\right)^2 \max\left(\frac{3}{2}\alpha_2^2 - \frac{1}{n+1}\alpha_1^2, \frac{1}{2} - \frac{1}{2}\alpha_1^2\right) = \overline{k}_1$$

2. $\theta_1 = (1,0), \theta_2 = (0,1)$

   If the players follow these strategies, they receive: $U_1 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_1^2, U_2 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2$. In order for this to be an equilibrium, there cannot be any profitable deviations. There are ten deviations we need to check.

   (i) Player 1 deviates to $\theta_1' = (0,1), x_1' = 0$: yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - \frac{n+1}{(n+2)^2}$

   (ii) Player 1 deviates to $\theta_1' = (0,0), x_1' = 0$: yields $U_1' = 0$

   (iii) Player 1 deviates to $\theta_1' = (1,0), x_i' = 1$: yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \max(\alpha_1^2, 1) - \frac{n+1}{(n+2)^2} - k$

   (iv) Player 1 deviates to $\theta_1' = (0,1), x_1' = 1$: yields $U_1' = 2\left(\frac{n+1}{n+2}\right)^2 - 2\frac{n+1}{(n+2)^2} - k$

   (v) Player 1 deviates to $\theta_1' = (0,0)$ and $x_1' = 1$: yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - \frac{n+1}{(n+2)^2} - k$

   (vi) Player 2 deviates to $\theta_2' = (1,0), x_2' = 0$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_2^2 - \frac{n+1}{(n+2)^2}\alpha_1^2$

   (vii) Player 2 deviates to $\theta_2' = (0,0), x_2' = 0$: yields $U_2' = 0$

   (viii) Player 2 deviates to $\theta_2' = (0,1), x_2' = 1$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \max(\alpha_2^2, 1) - \frac{n+1}{(n+2)^2}\alpha_1^2 - k$

   (ix) Player 2 deviates to $\theta_2' = (1,0), x_2' = 1$: yields $U_2' = 2\left(\frac{n+1}{n+2}\right)^2 \alpha_2^2 - 2\frac{n+1}{(n+2)^2}\alpha_1^2 - k$

   (x) Player 2 deviates to $\theta_2' = (0,0)$ and $x_2' = 1$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_2^2 - \frac{n+1}{(n+2)^2}\alpha_1^2 - k$

   Observe that (2) will be an equilibrium if and only if (i) through (x) are not profitable deviations. This yields ten conditions for (2) to be an equilibrium, but only conditions (i), (iii), (iv), (vi), and (ix) are binding:

Condition (i): $\alpha_1^2 \geq \frac{n-1}{n+1}$

Condition (iii): $k \geq -\left(\frac{n+1}{n+2}\right)^2 \left(\frac{1}{n+1}\right)$

Condition (iv): $k \geq \left(\frac{n+1}{n+2}\right)^2 \left(\frac{2n}{n+1} - \frac{1}{2}\alpha_1^2\right)$

Condition (vi): $\alpha_1^2 \geq \frac{n+1}{2}\left(\alpha_2^2 - 1\right)$

Condition (ix): $k \geq \left(\frac{n+1}{n+2}\right)^2 \left(2\alpha_2^2 - \frac{2}{n+1}\alpha_1^2 - \frac{1}{2}\right)$

If we combine conditions (iii), (vi), and (ix), we obtain:

$k \geq \left(\frac{n+1}{n+2}\right)^2 \max\left(\frac{2n}{n+1} - \frac{1}{2}\alpha_1^2, 2\alpha_2^2 - \frac{2}{n+1}\alpha_1^2 - \frac{1}{2}, \frac{-1}{n+1}\right) = \overline{k}_5$

3. $\theta_1 = (1,0), \theta_2 = (0,0)$

If the players follow these strategies, $U_2 = 0$. If player 2 deviates to $\theta_2' = (0,1)$, $x_2' = 0$, this yields: $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2$. So, there is always a profitable deviation. Thus, (3) is never an equilibrium.

4. $\theta_1 = \theta_2 = (0,1)$

If the players follow these strategies, $U_1 = U_2 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - \frac{n+1}{(n+2)^2}$. In order for this to be an equilibrium, there cannot be any profitable deviations. There are ten deviations we need to check.

(i) Player 1 deviates to $\theta_1' = (1,0)$, $x_1' = 0$: yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_1^2$

(ii) Player 1 deviates to $\theta_1' = (0,0)$, $x_1' = 0$: yields $U_1' = 0$

(iii) Player 1 deviates to $\theta_1' = (1,0)$, $x_i' = 1$: yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \max(\alpha_1^2, 1) - \frac{n+1}{(n+2)^2} - k$

(iv) Player 1 deviates to $\theta_1' = (0,1)$, $x_1' = 1$: yields $U_1' = 2\left(\frac{n+1}{n+2}\right)^2 - 2\frac{n+1}{(n+2)^2} - k$

(v) Player 1 deviates to $\theta_1' = (0,0)$ and $x_1' = 1$: yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - \frac{n+1}{(n+2)^2} - k$

(vi) Player 2 deviates to $\theta_2' = (1,0)$, $x_2' = 0$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_2^2$

(vii) Player 2 deviates to $\theta_2' = (0,0)$, $x_2' = 0$: yields $U_2' = 0$

(viii) Player 2 deviates to $\theta_2' = (0,1)$, $x_2' = 1$: yields $U_2' = 2\left(\frac{n+1}{n+2}\right)^2 - 2\frac{n+1}{(n+2)^2} - k$

(ix) Player 2 deviates to $\theta_2' = (1,0)$, $x_2' = 1$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \max(\alpha_2^2, 1) - \frac{n+1}{(n+2)^2} - k$

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(x) Player 2 deviates to $\theta_2' = (0,0)$ and $x_2' = 1$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - \frac{n+1}{(n+2)^2} - k$

Observe that (4) will be an equilibrium if and only if (i) through (x) are not profitable deviations. This yields ten conditions for (4) to be an equilibrium, but only conditions (iv/viii) and (vi) are binding:

Condition (iv/viii): $k \geq \left(\frac{n+1}{n+2}\right)^2\left(\frac{3}{2} - \frac{1}{n+1}\right) = \bar{k}_2$

Condition (vi): $\alpha_2^2 \leq \frac{n-1}{n+1}$

5. $\theta_1 = (0,1), \theta_2 = (1,0)$

If the players follow these strategies, $U_1 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2$, $U_2 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_2^2$.

In order for this to be an equilibrium, there cannot be any profitable deviations. There are ten deviations we need to check.

(i) Player 1 deviates to $\theta_1' = (1,0)$, $x_1' = 0$: yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2\alpha_1^2 - \frac{n+1}{(n+2)^2}\alpha_2^2$

(ii) Player 1 deviates to $\theta_1' = (0,0)$, $x_1' = 0$: yields $U_1' = 0$

(iii) Player 1 deviates to $\theta_1' = (1,0)$, $x_i' = 1$: yields $U_1' = 2\left(\frac{n+1}{n+2}\right)^2\alpha_1^2 - 2\frac{n+1}{(n+2)^2}\alpha_2^2 - k$

(iv) Player 1 deviates to $\theta_1' = (0,1)$, $x_1' = 1$: yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2\max(\alpha_1^2, 1) - \frac{n+1}{(n+2)^2}\alpha_2^2 - k$

(v) Player 1 deviates to $\theta_1' = (0,0)$ and $x_1' = 1$: yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2\alpha_1^2 - \frac{n+1}{(n+2)^2}\alpha_2^2 - k$

(vi) Player 2 deviates to $\theta_2' = (0,1)$, $x_2' = 0$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - \frac{n+1}{(n+2)^2}$

(vii) Player 2 deviates to $\theta_2' = (0,0)$, $x_2' = 0$: yields $U_2' = 0$

(viii) Player 2 deviates to $\theta_2' = (0,1)$, $x_2' = 1$: yields $U_2' = 2\left(\frac{n+1}{n+2}\right)^2 - 2\frac{n+1}{(n+2)^2} - k$

(ix) Player 2 deviates to $\theta_2' = (1,0)$, $x_2' = 1$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2\max(\alpha_2^2, 1) - \frac{n+1}{(n+2)^2} - k$

(x) Player 2 deviates to $\theta_2' = (0,0)$ and $x_2' = 1$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - \frac{n+1}{(n+2)^2} - k$

Observe that (5) will be an equilibrium if and only if (i) through (x) are not profitable deviations. This yields ten conditions for (5) to be an equilibrium, but only conditions (i), (iii), (v), (vi), (viii), and (ix) are binding:

Condition (i): $\alpha_1^2 \leq 1 + \frac{2}{n+1}\alpha_2^2$

Condition (iii): $k \geq \left(\frac{n+1}{n+2}\right)^2 \left(2\alpha_1^2 - \frac{2}{n+1}\alpha_2^2 - \frac{1}{2}\right)$

Condition (v): $k > \left(\frac{n+1}{n+2}\right)^2 \left(\frac{1}{2}\alpha_1^2 - \frac{1}{n+1}\alpha_2^2 - \frac{1}{2}\right) = \overline{k}_4$ (this inequality is strict because players prefer, all else equal, not to value activities)

Condition (vi): $\alpha_2^2 \geq \frac{n-1}{n+1}$

Condition (viii): $k \geq \left(\frac{n+1}{n+2}\right)^2 \left(\frac{2n}{n+1} - \frac{1}{2}\alpha_2^2\right)$

Condition (ix): $k \geq - \left(\frac{n+1}{n+2}\right)^2 \left(\frac{1}{n+1}\right)$

If we combine conditions (iii), (viii), and (ix), we obtain:

$k \geq \left(\frac{n+1}{n+2}\right)^2 \max\left(\frac{2n}{n+1} - \frac{1}{2}\alpha_2^2, 2\alpha_1^2 - \frac{2}{n+1}\alpha_2^2 - \frac{1}{2}, \frac{-1}{n+1}\right) = \overline{k}_3$

6. $\theta_1 = (0,1), \theta_2 = (0,0)$

   If the players follow these strategies, $U_2 = 0$. If player 2 deviates to $\theta_2' = (1,0)$, $x_2' = 0$, this yields: $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_2^2$. So, there is always a profitable deviation. Thus, (6) is never an equilibrium.

7. $\theta_1 = (0,0), \theta_2 = (1,0)$

   If the players follow these strategies, $U_1 = 0$. If player 1 deviates to $\theta_1' = (0,1)$, $x_i' = 0$, this yields: $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2$. So, there is always a profitable deviation. Thus, (7) is never an equilibrium.

8. $\theta_1 = (0,0), \theta_2 = (0,1)$

   If the players follow these strategies, $U_1 = 0$. If player 1 deviates to $\theta_1' = (1,0)$, $x_i' = 0$, this yields: $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_1^2$. So, there is always a profitable deviation. Thus, (8) is never an equilibrium.

9. $\theta_1 = \theta_2 = (0,0)$

   If the players follow these strategies, $U_1 = 0$. If player 1 deviates to $\theta_1' = (1,0)$, $x_i' = 0$, this yields: $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_1^2$. So, there is always a profitable deviation. Thus, (9) is never an equilibrium.

Case 2a: Interaction, initiated by both players ($x_1 = x_2 = 1$)

Observe that there will be a profitable deviation to $x_i' = 0$ if $k > 0$. If $k \leq 0$, deviating to $x_i' = 0$ is (weakly) unprofitable. Therefore, it will be sufficient, after imposing the condition $k \leq 0$, to restrict attention to deviations in which $x_i' = 1$.

There are nine types of combinations to check.

1. $\theta_1 = \theta_2 = (1, 0)$

   If the players follow these strategies, they receive: $U_1 = 2 \left(\frac{n+1}{n+2}\right)^2 \alpha_1^2 - 4 \frac{n+1}{(n+2)^2} \alpha_2^2 - k$, $U_2 = 2 \left(\frac{n+1}{n+2}\right)^2 \alpha_2^2 - 4 \frac{n+1}{(n+2)^2} \alpha_1^2 - k$. In order for this to be an equilibrium, there cannot be any profitable deviations. There are four deviations we need to check.

   (i) Player 1 deviates to $\theta_1' = (0, 1)$, $x_1' = 1$: yields $U_1' = \frac{1}{2} \left(\frac{n+1}{n+2}\right)^2 \max(\alpha_1^2, 1) - 2 \frac{n+1}{(n+2)^2} \alpha_2^2 - k$.

   (ii) Player 1 deviates to $\theta_1' = (0, 0)$, $x_1' = 1$: yields $U_1' = \frac{1}{2} \left(\frac{n+1}{n+2}\right)^2 \alpha_1^2 - 2 \frac{n+1}{(n+2)^2} \alpha_2^2 - k$.

   (iii) Player 2 deviates to $\theta_2' = (0, 1)$, $x_2' = 1$: yields $U_2' = \frac{1}{2} \left(\frac{n+1}{n+2}\right)^2 \max(\alpha_2^2, 1) - 2 \frac{n+1}{(n+2)^2} \alpha_1^2 - k$.

   (iv) Player 2 deviates to $\theta_2' = (0, 0)$, $x_2' = 1$: yields $U_2' = \frac{1}{2} \left(\frac{n+1}{n+2}\right)^2 \alpha_2^2 - 2 \frac{n+1}{(n+2)^2} \alpha_1^2 - k$.

   Observe that (1) will be an equilibrium if and only if (i) through (iv) are not profitable deviations and $k \leq 0$. Only conditions (i) and (ii) are binding:

   Condition (i): $\alpha_1^2 \geq \frac{1}{4} + \frac{1}{n+1} \alpha_2^2$. Observe that this condition fails to hold when $n = 0$.

   Condition (ii): $\alpha_1^2 > \frac{4}{3(n+1)} \alpha_2^2$ (this inequality is strict because players prefer not to value activities, all else equal).

   Condition (k): $k \leq 0$

2. $\theta_1 = (1, 0), \theta_2 = (0, 1)$

   There will be a profitable deviation for player 1 to $\theta_1' = (0, 0)$, $x_1' = 1$ if $\alpha_1^2 \leq 1$. There will be a profitable deviation for player 2 to $\theta_2' = (0, 0)$, $x_2' = 1$ if $\alpha_2^2 \geq 1$. So, we must have: $\alpha_1^2 > 1$ and $\alpha_2^2 < 1$. But, this is not possible given that $\alpha_2^2 \geq \alpha_1^2$. Thus, this combination will never be an equilibrium.

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3. $\theta_1 = (1,0), \theta_2 = (0,0)$

   If the players follow these strategies, $U_2 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2\alpha_2^2 - \frac{n+1}{(n+2)^2}\alpha_1^2 - k$. If player 2 deviates to $\theta_2' = (1,0)$, $x_2' = 1$, this yields: $U_2' = 2\left(\frac{n+1}{n+2}\right)^2\alpha_2^2 - 2\frac{n+1}{(n+2)^2}\alpha_1^2 - k$, which is profitable whenever $\frac{3}{2}\alpha_2^2 > \frac{1}{n+1}\alpha_1^2$ (which always holds). Thus, (3) is never an equilibrium.

4. $\theta_1 = \theta_2 = (0,1)$

   If the players follow these strategies, they receive: $U_1 = U_2 = 2\left(\frac{n+1}{n+2}\right)^2 - 4\frac{n+1}{(n+2)^2} - k$. In order for this to be an equilibrium, there cannot be any profitable deviations. There are four deviations we need to check.

   (i) Player 1 deviates to $\theta_1' = (1,0)$, $x_1' = 1$: yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2\max(\alpha_1^2, 1) - 2\frac{n+1}{(n+2)^2} - k$.

   (ii) Player 1 deviates to $\theta_1' = (0,0)$, $x_1' = 1$: yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - 2\frac{n+1}{(n+2)^2} - k$.

   (iii) Player 2 deviates to $\theta_2' = (1,0)$, $x_2' = 1$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2\max(\alpha_2^2, 1) - 2\frac{n+1}{(n+2)^2} - k$.

   (iv) Player 2 deviates to $\theta_2' = (0,0)$, $x_2' = 1$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - 2\frac{n+1}{(n+2)^2} - k$.

   Observe that (4) will be an equilibrium if and only if (i) through (iv) are not profitable deviations and $k \leq 0$. Only condition (iii) is binding:

   Condition (iii): $\frac{4n}{n+1} \geq \alpha_2^2$. Observe that this condition does not hold when $n = 0$.

   Condition (k): $k \leq 0$

5. $\theta_1 = (0,1), \theta_2 = (1,0)$

   There will be a profitable deviation for player 1 to $\theta_1' = (0,0)$, $x_1' = 1$ if $\alpha_1^2 \geq 1$. There will be a profitable deviation for player 2 to $\theta_2' = (0,0)$, $x_2' = 1$ if $\alpha_2^2 \leq 1$. So, we will assume for what follows that $\alpha_1^2 < 1$ and $\alpha_2^2 > 1$. If the players follow these strategies, they receive: $U_1 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - \frac{n+1}{(n+2)^2}\alpha_2^2 - k$, $U_2 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2\alpha_2^2 - \frac{n+1}{(n+2)^2} - k$. In order for this to be an equilibrium, there cannot be any profitable deviations. There are four deviations we need to check.

   (i) Player 1 deviates to $\theta_1' = (1,0)$, $x_1' = 1$: yields $U_1' = 2\left(\frac{n+1}{n+2}\right)^2\alpha_1^2 - 2\frac{n+1}{(n+2)^2}\alpha_2^2 - k$.

(ii) Player 1 deviates to $\theta'_1 = (0,0)$, $x'_1 = 1$: yields $U'_1 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_1^2 - \frac{n+1}{(n+2)^2}\alpha_2^2 - k$.

(iii) Player 2 deviates to $\theta'_2 = (0,1)$, $x'_2 = 1$: yields $U'_2 = 2\left(\frac{n+1}{n+2}\right)^2 - 2\frac{n+1}{(n+2)^2} - k$.

(iv) Player 2 deviates to $\theta'_2 = (0,0)$, $x'_2 = 1$: yields $U'_2 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - \frac{n+1}{(n+2)^2} - k$.

Observe that (5) will be an equilibrium if and only if (i) through (iv) are not profitable deviations and $k \leq 0$. Only conditions (i), (ii), and (iii) are binding:

Condition (i): $\alpha_2^2 \geq (n+1)\left(2\alpha_1^2 - \frac{1}{2}\right)$

Condition (ii): $1 > \alpha_1^2$

Condition (iii): $\alpha_2^2 \geq \frac{4n+2}{n+1}$. Observe that this condition is violated when $n \geq 2$.

Condition (k): $k \leq 0$

6. $\theta_1 = (0,1), \theta_2 = (0,0)$

If the players follow these strategies, $U_2 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - \frac{n+1}{(n+2)^2} - k$. If player 2 deviates to $\theta'_2 = (0,1)$, $x'_2 = 1$, this yields: $U'_2 = 2\left(\frac{n+1}{n+2}\right)^2 - 2\frac{n+1}{(n+2)^2} - k$, which is always profitable. Thus, (6) is never an equilibrium.

7. $\theta_1 = (0,0), \theta_2 = (1,0)$

If the players follow these strategies: $U_1 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_1^2 - \frac{n+1}{(n+2)^2}\alpha_2^2 - k$, $U_2 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_2^2 - \frac{n+1}{(n+2)^2}\alpha_1^2 - k$. In order for this to be an equilibrium, there cannot be any profitable deviations. There are four deviations we need to check.

(i) Player 1 deviates to $\theta'_1 = (1,0)$, $x'_1 = 1$: yields $U'_1 = 2\left(\frac{n+1}{n+2}\right)^2 \alpha_1^2 - 2\frac{n+1}{(n+2)^2}\alpha_2^2 - k$.

(ii) Player 1 deviates to $\theta'_1 = (0,1)$, $x'_1 = 1$: yields $U'_1 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \max(\alpha_1^2, 1) - \frac{n+1}{(n+2)^2}\alpha_2^2 - k$.

(iii) Player 2 deviates to $\theta'_2 = (0,1)$, $x'_2 = 1$: yields $U'_2 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - k$.

(iv) Player 2 deviates to $\theta'_2 = (0,0)$, $x'_2 = 1$: yields $U'_2 = -k$.

Observe that (7) will be an equilibrium if and only if (i) through (iv) are not profitable deviations and $k \leq 0$. This gives us the following conditions:

Condition (i): $\alpha_2^2 \geq \frac{3}{2}(n+1)\alpha_1^2$

Condition (ii): $\alpha_1^2 > 1$. Observe that the combination of conditions (i) and (ii) are violated for $n \geq 2$.

Condition (iii): $\alpha_2^2 \geq 1 + \frac{2}{n+1}\alpha_1^2$

Condition (k): $k \leq 0$

8. $\theta_1 = (0,0), \theta_2 = (0,1)$

   If the players follow these strategies, $U_1 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - \frac{n+1}{(n+2)^2} - k$. If player 1 deviates to $\theta_1' = (0,1)$, $x_1' = 1$, this yields: $U_1' = 2\left(\frac{n+1}{n+2}\right)^2 - 2\frac{n+1}{(n+2)^2} - k$, which is always profitable. Thus, (8) is never an equilibrium.

9. $\theta_1 = \theta_2 = (0,0)$

   If the players follow these strategies, $U_1 = -k$. If player 1 deviates to $\theta_1' = (0,1)$, $x_1' = 1$, this yields: $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - k$, which is always profitable. Thus, (9) is never an equilibrium.

Case 2b: Interaction, initiated by player 1 only ($x_1 = 1, x_2 = 0$)

Observe that there will be a profitable deviation to $x_2' = 1$ if $k < 0$. There will not be a profitable deviation to $x_2' = 1$ if $k \geq 0$. Therefore, it will be sufficient, after imposing the condition $k \geq 0$, to restrict attention to deviations in which $x_2' = 0$.

It is useful to observe that, if a particular (θ_1, θ_2) combination was never an equilibrium in Case 2a, it will never be an equilibrium in Case 2b. The reason is that there will be an equivalent profitable deviation in Case 2b to the one that existed in Case 2a. For this reason, there are only four types of (θ_1, θ_2) combinations that need to be checked:

1. $\theta_1 = \theta_2 = (1,0)$

   If the players follow these strategies, they receive: $U_1 = 2\left(\frac{n+1}{n+2}\right)^2 \alpha_1^2 - 4\frac{n+1}{(n+2)^2}\alpha_2^2 - k$, $U_2 = 2\left(\frac{n+1}{n+2}\right)^2 \alpha_2^2 - 4\frac{n+1}{(n+2)^2}\alpha_1^2$. In order for this to be an equilibrium, there cannot be any profitable deviations. There are seven types of deviations we need to check.

   (i) Player 1 deviates to $\theta_1' = (0,1)$, $x_1' = 1$: yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \max(\alpha_1^2, 1) - 2\frac{n+1}{(n+2)^2}\alpha_2^2 - k$.

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(ii) Player 1 deviates to $\theta'_1 = (0,0)$, $x'_1 = 1$: yields $U'_1 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_1^2 - 2\frac{n+1}{(n+2)^2}\alpha_2^2 - k$.

(iii) Player 1 deviates to $\theta'_1 = (1,0)$, $x'_1 = 0$: yields $U'_1 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_1^2 - 2\frac{n+1}{(n+2)^2}\alpha_2^2$.

(iv) Player 1 deviates to $\theta'_1 = (0,1)$, $x'_1 = 0$: yields $U'_1 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2$.

(v) Player 1 deviates to $\theta'_1 = (0,0)$, $x'_1 = 0$: yields $U'_1 = 0$.

(vi) Player 2 deviates to $\theta'_2 = (0,1)$, $x'_2 = 0$: yields $U'_2 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \max(\alpha_2^2, 1) - 2\frac{n+1}{(n+2)^2}\alpha_1^2$.

(vii) Player 2 deviates to $\theta'_2 = (0,0)$, $x'_2 = 0$: yields $U'_2 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_2^2 - 2\frac{n+1}{(n+2)^2}\alpha_1^2$.

Observe that (1) will be an equilibrium if and only if (i) through (vii) are not profitable deviations and $k \geq 0$. Only conditions (i), (ii), (iii), and (iv) are binding:

Condition (i): $\alpha_1^2 \geq \frac{1}{4} + \frac{1}{n+1}\alpha_2^2$

Condition (ii): $\alpha_1^2 > \frac{4}{3(n+1)}\alpha_2^2$. Observe that this condition fails when $n = 0$.

Condition (iii): $\left(\frac{n+1}{n+2}\right)^2 \left(\frac{3}{2}\alpha_1^2 - \frac{2}{n+1}\alpha_2^2\right) \geq k$

Condition (iv): $\left(\frac{n+1}{n+2}\right)^2 \left(2\alpha_1^2 - \frac{4}{n+1}\alpha_2^2 - \frac{1}{2}\right) \geq k$

Condition (k): $k \geq 0$

2. $\theta_1 = \theta_2 = (0,1)$

If the players follow these strategies, they receive: $U_1 = 2\left(\frac{n+1}{n+2}\right)^2 - 4\frac{n+1}{(n+2)^2} - k$, $U_2 = 2\left(\frac{n+1}{n+2}\right)^2 - 4\frac{n+1}{(n+2)^2}$. In order for this to be an equilibrium, there cannot be any profitable deviations. There are seven types of deviations we need to check.

(i) Player 1 deviates to $\theta'_1 = (1,0)$, $x'_1 = 1$: yields $U'_1 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \max(\alpha_1^2, 1) - 2\frac{n+1}{(n+2)^2} - k$.

(ii) Player 1 deviates to $\theta'_1 = (0,0)$, $x'_1 = 1$: yields $U'_1 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - 2\frac{n+1}{(n+2)^2} - k$.

(iii) Player 1 deviates to $\theta'_1 = (1,0)$, $x'_1 = 0$: yields $U'_1 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_1^2$.

(iv) Player 1 deviates to $\theta'_1 = (0,1)$, $x'_1 = 0$: yields $U'_1 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - 2\frac{n+1}{(n+2)^2}$.

(v) Player 1 deviates to $\theta'_1 = (0,0)$, $x'_1 = 0$: yields $U'_1 = 0$.

(vi) Player 2 deviates to $\theta'_2 = (1,0)$, $x'_2 = 0$: yields $U'_2 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \max(\alpha_2^2, 1) - 2\frac{n+1}{(n+2)^2}$.

(vii) Player 2 deviates to $\theta_2' = (0,0)$, $x_2' = 0$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - 2\frac{n+1}{(n+2)^2}$.

Observe that (2) will be an equilibrium if and only if (i) through (vii) are not profitable deviations and $k \geq 0$. Only conditions (iii), (iv), and (vi) are binding:

Condition (iii): $\left(\frac{n+1}{n+2}\right)^2 \left(2\left(\frac{n-1}{n+1}\right) - \frac{1}{2}\alpha_1^2\right) \geq k$

Condition (iv): $\left(\frac{n+1}{n+2}\right)^2 \left(\frac{3}{2} - \frac{2}{n+1}\right) \geq k$

Condition (vi): $\frac{4n}{n+1} \geq \alpha_2^2$. Observe that this condition is violated when $n = 0$.

Condition (k): $k \geq 0$

3. $\theta_1 = (0,1), \theta_2 = (1,0)$

There will be a profitable deviation for player 1 to $\theta_1' = (0,0)$, $x_1' = x_1$ if $\alpha_1^2 \geq 1$. There will be a profitable deviation for player 2 to $\theta_2' = (0,0)$, $x_2' = x_2$ if $\alpha_2^2 \leq 1$. So, we will assume for what follows that $\alpha_1^2 < 1$ and $\alpha_2^2 > 1$. If the players follow these strategies, player 1 receives: $U_1 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - \frac{n+1}{(n+2)^2}\alpha_2^2 - k$. Suppose player 1 deviates to $\theta_1' = (1,0)$, $x_1' = 0$. This deviation yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2$, which is profitable unless $-\frac{n+1}{(n+2)^2}\alpha_2^2 \geq k$. But, it is also required that $k \geq 0$ for there to be no profitable deviations. Since these inequalities cannot simultaneously hold, a profitable deviations always exists. Thus, (3) is never an equilibrium.

4. $\theta_1 = (0,0), \theta_2 = (1,0)$

If the players follow these strategies: $U_1 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2\alpha_1^2 - \frac{n+1}{(n+2)^2}\alpha_2^2 - k$, $U_2 = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2\alpha_2^2 - \frac{n+1}{(n+2)^2}\alpha_1^2$. Suppose player 1 deviates to $\theta_1' = (1,0)$, $x_1' = 1$, which yields $U_1' = 2\left(\frac{n+1}{n+2}\right)^2\alpha_1^2 - 2\frac{n+1}{(n+2)^2}\alpha_2^2 - k$. This will be a profitable deviation unless $\alpha_2^2 \geq \frac{3}{2}(n+1)\alpha_1^2$. Now, suppose player 1 deviates to $\theta_1' = (0,1)$, $x_1' = 1$, which yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2\max(\alpha_1^2, 1) - \frac{n+1}{(n+2)^2}\alpha_2^2 - k$. This deviation will be profitable unless $\alpha_1^2 > 1$. The conditions (i) $\alpha_2^2 \geq \frac{3}{2}(n+1)\alpha_1^2$ and (ii) $\alpha_1^2 > 1$ imply that $\alpha_2^2 > \frac{3}{2}(n+1)\alpha_1^2$, which is violated for all $n \geq 2$. So, no equilibria of type (4) exist when $n \geq 2$.

We will now show that no equilibria of type (4) exist when $n < 2$. Suppose player 1 deviates to $\theta_1' = (1,0)$, $x_1' = 0$. This yields: $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2\alpha_1^2 - \frac{n+1}{(n+2)^2}\alpha_2^2$. This deviation is profitable unless $k \leq 0$. But, we also need $k \geq 0$ (otherwise, there will be a profitable deviation). So, the only possible value of k is $k = 0$. Finally, suppose player 1 deviates to $\theta_1' = (0,1)$, $x_1' = 0$, which yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2$. This

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deviation is profitable unless $\frac{1}{2}\left(\frac{n+1}{n+2}\right)^2\alpha_1^2 - \frac{n+1}{(n+2)^2}\alpha_2^2 - k \geq \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2$. Combining this with the assumption that $k = 0$, we find that it is necessary to have: $\alpha_1^2 \geq \frac{2}{n+1}\alpha_2^2 + 1$. But, when $n < 2$, this condition is always violated. Thus, no equilibrium of type (4) exists.

Case 2c: Interaction, initiated by player 2 only ($x_1 = 0, x_2 = 1$)

Observe that there will be a profitable deviation to $x_1' = 1$ if $k < 0$. There will not be a profitable deviation to $x_1' = 1$ if $k \geq 0$. Therefore, it will be sufficient, after imposing the condition $k \geq 0$, to restrict attention to deviations in which $x_1' = 0$.

It is useful to observe that, if a particular (θ_1, θ_2) combination was never an equilibrium in Case 2a, it will never be an equilibrium in Case 2c. The reason is that there will be an equivalent profitable deviation in Case 2c to the one that existed in Case 2a. For this reason, there are only four types of (θ_1, θ_2) combinations that need to be checked.

1. $\theta_1 = \theta_2 = (1, 0)$

   If the players follow these strategies, they receive: $U_1 = 2\left(\frac{n+1}{n+2}\right)^2\alpha_1^2 - 4\frac{n+1}{(n+2)^2}\alpha_2^2$, $U_2 = 2\left(\frac{n+1}{n+2}\right)^2\alpha_2^2 - 4\frac{n+1}{(n+2)^2}\alpha_1^2 - k$. In order for this to be an equilibrium, there cannot be any profitable deviations. There are seven types of deviations we need to check.

   (i) Player 2 deviates to $\theta_2' = (0, 1)$, $x_2' = 1$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2\max(\alpha_2^2, 1) - 2\frac{n+1}{(n+2)^2}\alpha_1^2 - k$.

   (ii) Player 2 deviates to $\theta_2' = (0, 0)$, $x_2' = 1$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2\alpha_2^2 - 2\frac{n+1}{(n+2)^2}\alpha_1^2 - k$.

   (iii) Player 2 deviates to $\theta_2' = (1, 0)$, $x_2' = 0$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2\alpha_2^2 - 2\frac{n+1}{(n+2)^2}\alpha_1^2$.

   (iv) Player 2 deviates to $\theta_2' = (0, 1)$, $x_2' = 0$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2$.

   (v) Player 2 deviates to $\theta_2' = (0, 0)$, $x_2' = 0$: yields $U_2' = 0$.

   (vi) Player 1 deviates to $\theta_1' = (0, 1)$, $x_1' = 0$: yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2\max(\alpha_1^2, 1) - 2\frac{n+1}{(n+2)^2}\alpha_2^2$.

   (vii) Player 1 deviates to $\theta_1' = (0, 0)$, $x_1' = 0$: yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2\alpha_1^2 - 2\frac{n+1}{(n+2)^2}\alpha_2^2$.

Observe that (1) will be an equilibrium if and only if (i) through (vii) are not profitable deviations and $k \geq 0$. Only conditions (iii), (iv), (vi), and (vii) are binding:

Condition (iii): $\left(\frac{n+1}{n+2}\right)^2 \left(\frac{3}{2}\alpha_2^2 - \frac{2}{n+1}\alpha_1^2\right) \geq k$

Condition (iv): $\left(\frac{n+1}{n+2}\right)^2 \left(2\alpha_2^2 - \frac{4}{n+1}\alpha_1^2 - \frac{1}{2}\right) \geq k$

Condition (vi): $\alpha_1^2 \geq \frac{1}{4} + \frac{1}{n+1}\alpha_2^2$

Condition (vii): $\frac{3}{4}\alpha_1^2 > \frac{1}{n+1}\alpha_2^2$. Observe that this condition fails to hold when $n = 0$.

Condition (k): $k \geq 0$

2. $\theta_1 = \theta_2 = (0, 1)$

   If the players follow these strategies, they receive: $U_1 = 2\left(\frac{n+1}{n+2}\right)^2 - 4\frac{n+1}{(n+2)^2}$, $U_2 = 2\left(\frac{n+1}{n+2}\right)^2 - 4\frac{n+1}{(n+2)^2} - k$. In order for this to be an equilibrium, there cannot be any profitable deviations. There are seven types of deviations we need to check.

   (i) Player 2 deviates to $\theta_2' = (1, 0)$, $x_2' = 1$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \max(\alpha_2^2, 1) - 2\frac{n+1}{(n+2)^2} - k$.

   (ii) Player 2 deviates to $\theta_2' = (0, 0)$, $x_2' = 1$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - 2\frac{n+1}{(n+2)^2} - k$.

   (iii) Player 2 deviates to $\theta_2' = (1, 0)$, $x_2' = 0$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_2^2$.

   (iv) Player 2 deviates to $\theta_2' = (0, 1)$, $x_2' = 0$: yields $U_2' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - 2\frac{n+1}{(n+2)^2}$.

   (v) Player 2 deviates to $\theta_2' = (0, 0)$, $x_2' = 0$: yields $U_2' = 0$.

   (vi) Player 1 deviates to $\theta_1' = (1, 0)$, $x_1' = 0$: yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \max(\alpha_1^2, 1) - 2\frac{n+1}{(n+2)^2}$.

   (vii) Player 1 deviates to $\theta_1' = (0, 0)$, $x_1' = 0$: yields $U_1' = \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 - 2\frac{n+1}{(n+2)^2}$.

Observe that (2) will be an equilibrium if and only if (i) through (vii) are not profitable deviations and $k \geq 0$. Only conditions (i), (iii), and (iv) are binding:

Condition (i): $\frac{4n}{n+1} \geq \alpha_2^2$. Observe that this condition is violated when $n = 0$.

Condition (iii): $\left(\frac{n+1}{n+2}\right)^2 \left(2\left(\frac{n-1}{n+1}\right) - \frac{1}{2}\alpha_2^2\right) \geq k$

Condition (iv): $\left(\frac{n+1}{n+2}\right)^2 \left(\frac{3}{2} - \frac{2}{n+1}\right) \geq k$

Condition (k): $k \geq 0$

3. $\theta_1 = (0, 1), \theta_2 = (1, 0)$

There will be a profitable deviation for player 1 to $\theta_1' = (0, 0)$, $x_1' = x_1$ if $\alpha_1^2 \geq 1$. There will be a profitable deviation for player 2 to $\theta_2' = (0, 0)$, $x_2' = x_2$ if $\alpha_2^2 \leq 1$. So, we will assume for what follows that $\alpha_1^2 < 1$ and $\alpha_2^2 > 1$. If the players follow these strategies, player 2 receives: $\frac{1}{2} \left(\frac{n+1}{n+2} \right)^2 \alpha_2^2 - \frac{n+1}{(n+2)^2} - k$. Suppose player 2 deviates to $\theta_2' = (1, 0)$, $x_2' = 0$. This deviation yields $U_2' = \frac{1}{2} \left(\frac{n+1}{n+2} \right)^2 \alpha_2^2$, which is profitable unless $-\frac{n+1}{(n+2)^2} \geq k$. But, it is also required that $k \geq 0$ for there to be no profitable deviations. Since these inequalities cannot simultaneously hold, a profitable deviations always exists. Thus, (3) is never an equilibrium.

4. $\theta_1 = (0, 0), \theta_2 = (1, 0)$

If the players follow these strategies: $U_1 = \frac{1}{2} \left(\frac{n+1}{n+2} \right)^2 \alpha_1^2 - \frac{n+1}{(n+2)^2} \alpha_2^2$, $U_2 = \frac{1}{2} \left(\frac{n+1}{n+2} \right)^2 \alpha_2^2 - \frac{n+1}{(n+2)^2} \alpha_1^2 - k$. In order for this to be an equilibrium, there cannot be any profitable deviations. There are seven deviations we need to check.

(i) Player 1 deviates to $\theta_1' = (1, 0)$, $x_1' = 0$: yields $U_1' = 2 \left(\frac{n+1}{n+2} \right)^2 \alpha_1^2 - 2 \frac{n+1}{(n+2)^2} \alpha_2^2$.

(ii) Player 1 deviates to $\theta_1' = (0, 1)$, $x_1' = 0$: yields $U_1' = \frac{1}{2} \left(\frac{n+1}{n+2} \right)^2 \max(\alpha_1^2, 1) - \frac{n+1}{(n+2)^2} \alpha_2^2$.

(iii) Player 2 deviates to $\theta_2' = (0, 1)$, $x_2' = 1$: yields $U_2' = \frac{1}{2} \left(\frac{n+1}{n+2} \right)^2 - k$.

(iv) Player 2 deviates to $\theta_2' = (0, 0)$, $x_2' = 1$: yields $U_2' = -k$.

(v) Player 2 deviates to $\theta_2' = (1, 0)$, $x_2' = 0$: yields $U_2' = \frac{1}{2} \left(\frac{n+1}{n+2} \right)^2 \alpha_2^2 - \frac{n+1}{(n+2)^2} \alpha_1^2$.

(vi) Player 2 deviates to $\theta_2' = (0, 1)$, $x_2' = 0$: yields $U_2' = \frac{1}{2} \left(\frac{n+1}{n+2} \right)^2$.

(vii) Player 2 deviates to $\theta_2' = (0, 0)$, $x_2' = 0$: yields $U_2' = 0$.

Observe that (4) will be an equilibrium if and only if (i) through (vii) are not profitable deviations and $k \leq 0$. Only conditions (i), (ii), (iii), (v), and (vi) will be binding:

Condition (i): $\alpha_2^2 \geq \frac{3}{2}(n + 1)\alpha_1^2$

Condition (ii): $\alpha_1^2 > 1$. Observe that the combination of conditions (i) and (ii) are violated for $n \geq 2$.

Condition (iii): $\alpha_2^2 \geq 1 + \frac{2}{n+1}\alpha_1^2$

Condition (v): $k \leq 0$. Observe that, combined with condition (k), this implies $k = 0$.

Condition (vi): $\frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \alpha_2^2 - \frac{n+1}{(n+2)^2}\alpha_1^2 - \frac{1}{2}\left(\frac{n+1}{n+2}\right)^2 \geq k$.

Given that $k = 0$, this condition can be rewritten as: $\alpha_2^2 \geq 1 + \frac{2}{n+1}\alpha_1^2$

Condition (k): $k \geq 0$

Case 2: Combined

If we combine Cases 2a, 2b, and 2c, we find the following.

1. An equilibrium with $\theta_1 = \theta_2 = (1,0)$ and interaction between the players exists if:

   (i) $\alpha_1^2 \geq \frac{1}{4} + \frac{1}{n+1}\alpha_2^2$, $\alpha_1^2 > \frac{4}{3(n+1)}\alpha_2^2$

   (ii) $k \leq \max\left(0, \left(\frac{n+1}{n+2}\right)^2 \min\left(\frac{3}{2}\alpha_2^2 - \frac{2}{n+1}\alpha_1^2, 2\alpha_2^2 - \frac{4}{n+1}\alpha_1^2 - \frac{1}{2}\right)\right) = \overline{k}_6$

   (iii) $n \geq 1$

2. An equilibrium with $\theta_1 = \theta_2 = (0,1)$ and interaction between the players exists if:

   (i) $\alpha_2^2 \leq \frac{4n}{n+1}$

   (ii) $k \leq \max\left(0, \left(\frac{n+1}{n+2}\right)^2 \min\left(2\left(\frac{n-1}{n+1}\right) - \frac{1}{2}\alpha_1^2, \frac{3}{2} - \frac{2}{n+1}\right)\right) = \overline{k}_7$

   (iii) $n \geq 1$

3. An equilibrium with $\theta_1 = (0,1)$, $\theta_2 = (1,0)$, and interaction between the players exists if:

   (i) $\alpha_2^2 \geq \max\left(\frac{4n+2}{n+1}, (n+1)\left(2\alpha_1^2 - \frac{1}{2}\right)\right)$.

   (ii) $\alpha_1^2 < 1$

   (iii) $k \leq 0$

   Observe that these conditions will be violated when $n \geq 2$.

4. An equilibrium with $\theta_1 = (0,0)$, $\theta_2 = (1,0)$, and interaction between the players exists if:

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(i) $\alpha_2^2 \geq \max(\frac{3}{2}(n+1)\alpha_1^2, 1 + \frac{2}{n+1}\alpha_1^2)$

(ii) $\alpha_1^2 > 1$

(iii) $k \leq 0$

Observe that these conditions will be violated when $n \geq 2$.

This completes the proof. □

Proof of Lemma 5. To prove the lemma, it is sufficient to show two things to be true. Holding α_1 and α_2 fixed:

(i) If an equilibrium exists for some value of k in which players interact and focus on different activities, no equilibrium exists for any value of k in which players focus on the same activity.

(ii) If an equilibrium exists for some value of k in which players interact and hold different values, no equilibrium exists for any value of k in which players hold the same values.

Let us assume, without loss of generality, that $\alpha_2 \geq \alpha_1$.

Case 1: $\alpha_2 > \overline{\alpha}_H$

Applying Proposition 2, (i) and (ii) can be restated as follows:

(i) If equilibria of type (3) exist for some k, equilibria of types (2), (4), and (5) do not exist for any values of k.

(ii) If equilibria of types (3) or (5) exist for some k, equilibria of types (2) and (4) do not exist for any values of k.

Showing (i): If an equilibrium of type (3) exists for some k, we must have $\alpha_1^2 \leq \frac{1}{4} + \frac{1}{2(n+1)}\alpha_2^2$ and $\alpha_1 < 1$. Let us suppose this to be the case. In order for an equilibrium of type (2) to exist for some k, we must have $\alpha_1^2 \geq 1 + \frac{2}{(n+1)}\alpha_2^2$. But, in order for this to hold, $\frac{1}{4} + \frac{1}{2(n+1)}\alpha_2^2 \geq \alpha_1^2 \geq 1 + \frac{2}{(n+1)}\alpha_2^2$, which can never be satisfied. Thus, equilibria of type (2) do not exist for any values of k. In order for an equilibrium of type (4) to exist for some k, we must have $\alpha_1^2 \geq \frac{1}{4} + \frac{1}{n+1}\alpha_2^2$. It follows that we must have $\frac{1}{4} + \frac{1}{n+1}\alpha_2^2 \leq \alpha_1^2 \leq \frac{1}{4} + \frac{1}{2(n+1)}\alpha_2^2$, which can never be satisfied. Thus, equilibria of type (4) do not exist for any values of k. In order for an equilibrium of type (5) to exist for some k, we must have $1 < \alpha_1^2$. But, this contradicts our assumption that $\alpha_1 < 1$. So, equilibria of type (5) do not exist for any values of k. Thus, (i) holds.

Showing (ii): We have already shown that, if an equilibrium of type (3) exists for some k, equilibria of types (2) and (4) do not exist for any k. If an equilibrium of type (5) exists for some k, $1 < \alpha_1^2 < \frac{2}{3(n+1)}\alpha_2^2$. Let us suppose that this condition holds. In order for an equilibrium of type (2) to exist, we must have $\alpha_1^2 \geq 1 + \frac{2}{(n+1)}\alpha_2^2$. It follows that we must have $\frac{2}{3(n+1)}\alpha_2^2 > 1 + \frac{2}{(n+1)}\alpha_2^2$, or $-\frac{4}{3(n+1)}\alpha_2^2 > 1$ (which can never be true). Thus, equilibria of type (2) cannot exist for any k. In order for an equilibrium of type (4) to exist, we must have $\alpha_1^2 > \frac{4}{3(n+1)}\alpha_2^2$. But, this contradicts our assumption that $\alpha_1^2 < \frac{2}{3(n+1)}\alpha_2^2$. So, equilibria of type (4) do not exist for any k. This shows that (ii) holds.

Case 2: $\alpha_1 < \overline{\alpha}_L$

Applying Proposition 3, (i) and (ii) can be restated as follows:

(i/ii) If equilibria of type (4) exist for some k, equilibria of types (1) and (3) do not exist for any values of k.

Showing (i/ii): If an equilibrium of type (4) exists for some k, we must have $\alpha_2^2 \geq 4 - \frac{2}{n+1}$. Let us suppose this to be the case. If an equilibrium of type (1) exists for some k, we must have $\alpha_2^2 \leq \frac{n-1}{n+1}$. But, this cannot hold since $\alpha_2^2 \geq 4 - \frac{2}{n+1}$ and $4 - \frac{2}{n+1} > \frac{n-1}{n+1}$. If an equilibrium of type (3) exists for some k, we must have $\alpha_2^2 \leq \frac{4n}{n+1}$. But, again, this cannot hold since $\alpha_2^2 \geq 4 - \frac{2}{n+1}$ and $4 - \frac{2}{n+1} > \frac{4n}{n+1}$. This establishes (i/ii).

Case 3: $\overline{\alpha}_L \leq \alpha_1, \alpha_2 \leq \overline{\alpha}_H$

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Applying Proposition 4, (i) and (ii) can be restated as follows:

(i) If equilibria of type (7) exist for some k, equilibria of types (1) and (2) do not exist for any values of k.

(ii) If equilibria of types (7) and (8) exist for some k, equilibria of types (1) and (2) do not exist for any values of k.

Showing (i): If an equilibrium of type (7) exists for some k, we must have $n \leq 1$. Suppose this is the case. An equilibrium of type (1) only exists if $\alpha_1^2 \geq 1 + \frac{2}{n+1}\alpha_2^2$. But since $n \leq 1$ and $\alpha_2 \geq \alpha_1$, this condition cannot be satisfied. Therefore, an equilibrium of type (1) cannot exist. An equilibrium of type (2) only exists if $\alpha_2^2 \leq \frac{n-1}{n+1}$. But, once again, since $n \leq 1$ and $\alpha_2 > 0$, this condition cannot be satisfied. Therefore, an equilibrium of type (2) cannot exist. This establishes (i).

Showing (ii): We have already shown that, if an equilibrium of type (7) exists for some k, equilibria of types (1) and (2) do not exist for any k. Existence of an equilibrium of type (8) also requires $n \leq 1$. Therefore, by the same logic, if an equilibrium of type (8) exists for some k, equilibria of types (1) and (2) do not exist for any k. This establishes (ii) and completes the proof. □

Proof of Proposition 5 and Lemma 6. First, let us establish that, in any equilibrium, players value one activity (except possibly a set of players with 0 mass).

As in the baseline model, the returns to effort will be (weakly) greater at one of the activities. Furthermore, players prefer, all else equal, not to exert effort at activities. It follows that players will exert effort at – at most – one activity in equilibrium. Players will either be below average or average at activities when they exert no effort. Since players prefer, all else equal, not to value activities, it follows that players will either value one activity or zero activities in equilibrium.

Suppose there is an equilibrium in which a fraction λ_s of the players value activity s. By a logic identical to that given in Lemma 5, players who value activity s will interact in equilibrium with all players who hold the same values (except perhaps a set of measure 0); players who value activity s will not interact with players who hold different values

(except perhaps a set of measure 0). Following a logic similar to that from Lemma 2, players valuing activity s will choose to exert effort $\lambda_s + \beta$ at activity s, and zero effort at other activities. Since a fraction λ_s of the population exerts effort $\lambda_s + \beta$ at activity s, and the rest of the population exerts effort 0 at activity s, the average achievement at activity s is: $\bar{a}_s = \lambda_s (\lambda_s + \beta)$. From this, we conclude that players who value activity s receive utility: $\left(\frac{1}{2} - \lambda_s\right)(\lambda_s + \beta)^2$.

Suppose now that a positive mass of players values no activity in equilibrium. (We will show that this creates a contradiction). By an identical logic to that given in the proof of Lemma 5, it follows that such players will not interact with any other player in equilibrium. And, by an identical logic to that given in Lemma 1, it follows that these players will choose to exert zero effort at all activities and will receive utility 0. If these players deviate and instead choose to value activity s, they would receive utility $\left(\frac{1}{2} - \lambda_s\right)(\lambda_s + \beta)^2$. Since this deviation cannot be profitable, we must have $\left(\frac{1}{2} - \lambda_s\right)(\lambda_s + \beta)^2 \leq 0$ for all s, which is true if and only if $\lambda_s \geq \frac{1}{2}$ for all s. The only way we can have $\lambda_s \geq \frac{1}{2}$ for all s is if $M = 2$ and $\lambda_1 = \lambda_2 = \frac{1}{2}$. But, if $\lambda_1 = \lambda_2 = \frac{1}{2}$, it follows that there cannot be a positive mass of players that value no activity (a contradiction).

Thus, we have established that, in any equilibrium, players values one activity (except possibly a set of players with 0 mass). Furthermore, we have established that players who value activity s receive utility $\left(\frac{1}{2} - \lambda_s\right)(\lambda_s + \beta)^2$ in equilibrium, where λ_s denotes the fraction of players who value activity s. We will refer to players who value activity s as a group of size λ_s. We will focus on characterizing the set of equilibria with the property that all groups are of equal size (i.e., for all s, $\lambda_s = \bar{\lambda}$ or 0).

Case 1: every activity is valued by a positive mass of players in equilibrium.

First, we will examine the possibility that equilibria exist in which every activity is valued by a positive mass of players. Since $\lambda_s > 0$ for all s, we must have $\lambda_s = \frac{1}{M}$ for all s.

Let us check whether it is an equilibrium for all activities to be valued by a fraction $\frac{1}{M}$ of the players. There are two ways in which players can deviate. One way in which a player can deviate is by joining a different group. But, since groups are all of the same size, it is not profitable to join a different group. A second way in which

a player can deviate is by choosing to value no activity, which yields utility 0. We know this is an unprofitable deviation since players who belong to groups receive utility $\left(\frac{1}{2} - \frac{1}{M}\right)\left(\frac{1}{M} + \beta\right)^2 \geq 0$. Hence, it is always an equilibrium for the players to divide into M groups of equal size.

Case 2: $m < M$ activities are valued by a positive mass of players in equilibrium.

Now, let us examine whether equilibria exist in which $m < M$ activities are valued by a positive mass of players. We must have $\lambda_s = \frac{1}{m}$ for all activities valued by a positive mass of players.

Let us check whether it is an equilibrium for a fraction $\frac{1}{m}$ of the population to value each of m activities. There are three ways in which players can deviate. One way in which a player can deviate is by joining a different group. But, since groups are all of the same size, it is not profitable to join a different group. A second way in which a player can deviate is by choosing to value no activity, which yields utility 0. A third way in which a player can deviate is by starting a new group (a group of size 0), which yields utility $\frac{1}{2}\beta^2$. Observe that this third type of deviation is always preferred to the second type of deviation. This deviation will be unprofitable if and only if: $\left(\frac{1}{2} - \frac{1}{m}\right)\left(\frac{1}{m} + \beta\right)^2 > \frac{1}{2}\beta^2$. The reason this inequality is strict is that a player who deviates forgoes having to pay a positive (but negligible) cost of initiating interaction with other players in his group. If $\beta \geq 1$, this deviation is profitable for all m. If $\beta < 1$, the condition can be rewritten as: $m > \overline{m}$, where $\overline{m}$ solves: $\left(\frac{1}{2} - \frac{1}{\overline{m}}\right)\left(\frac{1}{\overline{m}} + \beta\right)^2 = \frac{1}{2}\beta^2$.

This completes the proof. $\qquad\square$